# 정적분 100題

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## (1) 각변환

$\sin(-\theta)=-\sin\theta$
$\sin(\pi+\theta)=-\sin\theta$
$\sin(\pi-\theta)=\sin\theta$
$\sin\left(\frac{\pi}{2}+\theta\right)=\cos\theta$
$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta$
$\sin\left(\frac{3}{2}\pi+\theta\right)=-\cos\theta$
$\sin\left(\frac{3}{2}\pi-\theta\right)=-\cos\theta$

$\cos(-\theta)=\cos\theta$
$\cos(\pi+\theta)=-\cos\theta$
$\cos(\pi-\theta)=-\cos\theta$
$\cos\left(\frac{\pi}{2}+\theta\right)=-\sin\theta$
$\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$
$\cos\left(\frac{3}{2}\pi+\theta\right)=\sin\theta$
$\cos\left(\frac{3}{2}\pi-\theta\right)=-\sin\theta$

$\tan(-\theta)=-\tan\theta$
$\tan(\pi+\theta)=\tan\theta$
$\tan(\pi-\theta)=-\tan\theta$
$\tan\left(\frac{\pi}{2}+\theta\right)=-\cot\theta$
$\tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta$
$\tan\left(\frac{3}{2}\pi+\theta\right)=-\cot\theta$
$\tan\left(\frac{3}{2}\pi-\theta\right)=\cot\theta$

## (2) 덧셈정리

$\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta$
$\cos(\alpha\pm\beta)=\cos\alpha\cos\beta\mp\sin\alpha\sin\beta$
$\tan(\alpha\pm\beta)=\dfrac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}\quad(1\mp\tan\alpha\tan\beta\neq 0)$

## (3) 배각공식

$\sin2\alpha=2\sin\alpha\cos\alpha$
$\cos2\alpha=\cos^2\alpha-\sin^2\alpha=2\cos^2\alpha-1=1-2\sin^2\alpha$
$\tan2\alpha=\dfrac{2\tan\alpha}{1-\tan^2\alpha}\quad(1-\tan^2\alpha\neq 0)$
$\sin3\alpha=3\sin\alpha-4\sin^3\alpha$
$\cos3\alpha=4\cos^3\alpha-3\cos\alpha$
$\tan3\alpha=\dfrac{3\tan\alpha-\tan^3\alpha}{1-3\tan^2\alpha}\quad(1-3\tan^2\alpha\neq 0)$

## (4) 반각공식

$\sin^2\dfrac{\alpha}{2}=\dfrac{1-\cos\alpha}{2}$
$\cos^2\dfrac{\alpha}{2}=\dfrac{1+\cos\alpha}{2}$
$\tan^2\dfrac{\alpha}{2}=\dfrac{1-\cos\alpha}{1+\cos\alpha}$

### (5) 곱을 합 또는 차로 고치는 공식

$\sin\alpha\cos\beta = \frac{1}{2}\{\sin(\alpha+\beta)+\sin(\alpha-\beta)\}$

$\cos\alpha\sin\beta = \frac{1}{2}\{\sin(\alpha+\beta)-\sin(\alpha-\beta)\}$

$\cos\alpha\cos\beta = \frac{1}{2}\{\cos(\alpha+\beta)+\cos(\alpha-\beta)\}$

$\sin\alpha\sin\beta = -\frac{1}{2}\{\cos(\alpha+\beta)-\cos(\alpha-\beta)\}$

### (6) 합 또는 차를 곱으로 고치는 공식

$\sin A+\sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

$\sin A-\sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

$\cos A+\cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

$\cos A-\cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

### (7) 삼각함수의 합성

$a\sin\theta+b\cos\theta = \sqrt{a^2+b^2}\sin(\theta+\alpha) = \sqrt{a^2+b^2}\cos(\theta-\beta)$

$\alpha = \tan^{-1}\left(\frac{b}{a}\right),\ \beta = \tan^{-1}\left(\frac{a}{b}\right) = \frac{\pi}{2}-\alpha$

### (8) 삼각함수 항등식

$\sin^2\theta+\cos^2\theta = 1$

$\tan^2\theta+1 = \sec^2\theta$

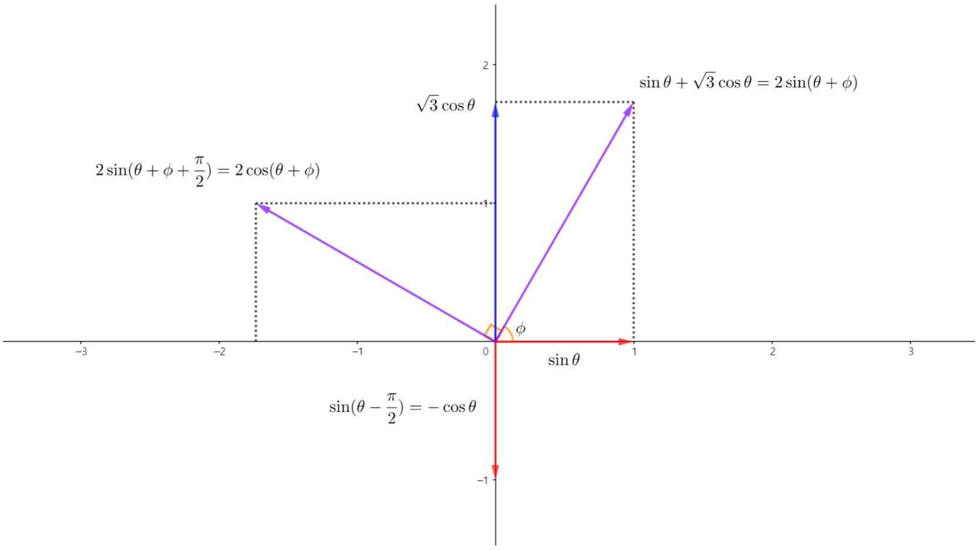
$\cot^2\theta+1 = \csc^2\theta$

### (9) 부정적분과 미분

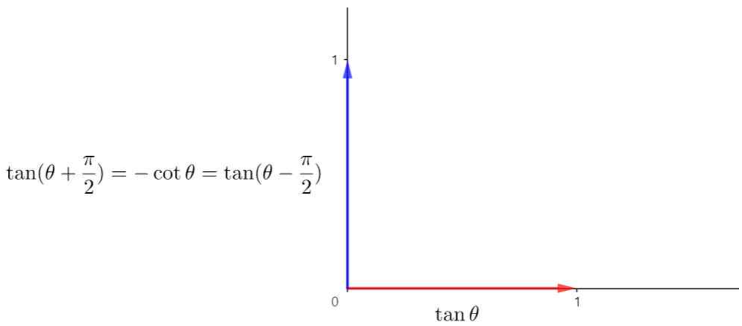
① $\frac{d}{dx}\left(\int f(x)dx\right) = f(x)$

② $\int\left(\frac{d}{dx}f(x)\right)dx = f(x)+C$

## (10) 삼각함수 위상자(Phasor)



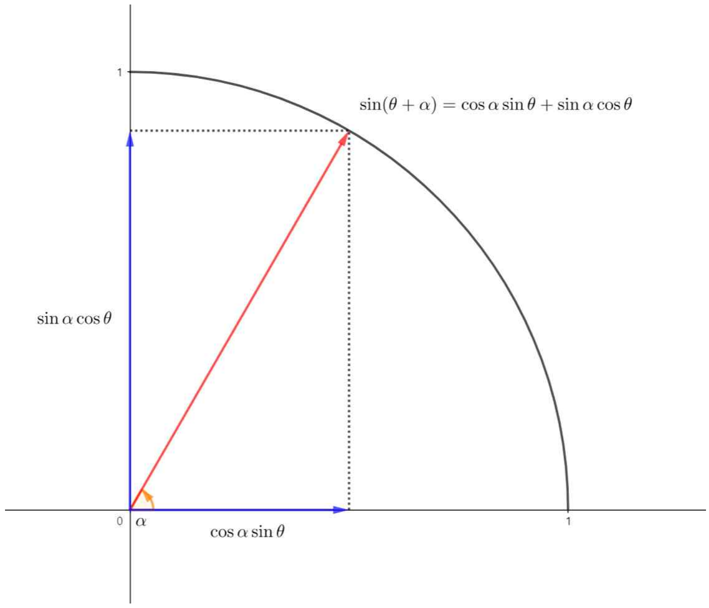

- $x$축의 양의 방향을 $\sin$축, $y$축의 양의 방향을 $\cos$축으로 설정하여 삼각함수의 위상을 벡터로 표현한다.
- 각이 더해질 경우 위상자는 길이는 유지된 채 반시계방향으로 회전한다.
- 위상자가 표현하는 삼각함수의 계수는 그 위상자의 길이로 표현된다.
- 서로 다른 두 위상자를 벡터합하면 이는 각 위상자가 표현하는 삼각함수의 합성과 같다. 가령, 위 사진에서 $\sin\theta + \sqrt{3}\cos\theta = 2\sin(\theta+\phi)$이고 $\phi = \frac{\pi}{3}$이다.
- $\sin\theta$ 위상자를 시계방향으로 $\frac{\pi}{2}$ 만큼 회전시키면 $\sin\left(\theta - \frac{\pi}{2}\right)$이며, 이는 $-\cos$축이므로 $\sin\left(\theta - \frac{\pi}{2}\right) = -\cos\theta$이다.
- $\sin$을 $\csc$로, $\cos$을 $\sec$로 바꾸면 $\csc$와 $\sec$에 대한 각변환이 가능하나 덧셈정리와 합성은 성립하지 않는다. 즉, $\csc\theta + \sqrt{3}\sec\theta = 2\csc(\theta+\phi)$는 성립하지 않는다.



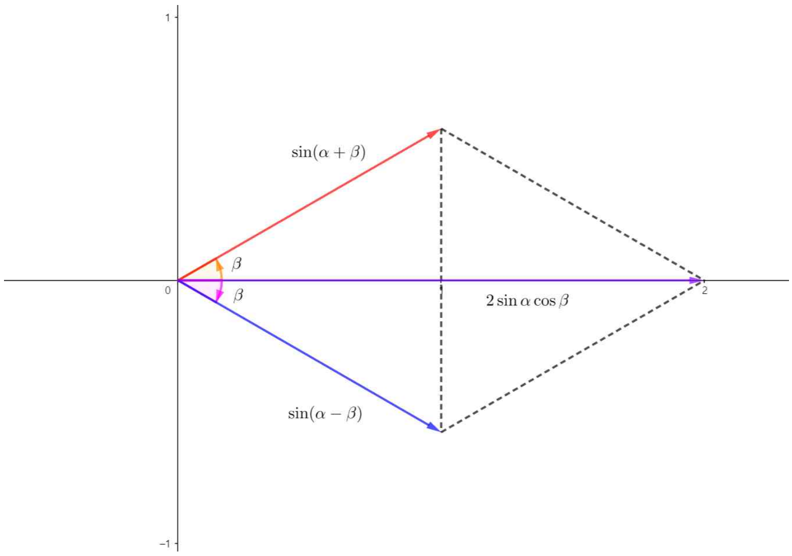

- $\tan$와 $-\cot$의 경우 위와 같이 $xy$평면의 제 1사분면만을 이용하여 도시할 수 있다. 이 경우 주기가 $\pi$이므로 시계방향으로 회전하는 경우 돌아간 각도를 $\frac{3}{2}\pi$가 아닌 $\frac{\pi}{2}$로 본다.

## (11) 삼각함수 위상자의 활용

- 삼각함수 위상자를 사용하면 (1), (2), (5), (6), (7)의 공식들은 모두 증명 가능하다. 이에 대한 예시로 (2)와 (5)의 첫 번째 공식의 증명을 제시해 놓는다.




그림과 같은 단위원에서 $\sin(\theta+\alpha)$는 $\sin\theta$축에서 길이 1인 위상자가 각도 $\alpha$만큼 회전한 것이다. 따라서 위상자의 종점에서 sin축, cos축에 각각 수선의 발을 내리면 원점을 시점으로 하고 두 수선의 발을 종점으로 하는 두 위상자의 길이는 각각 $\cos\alpha$, $\sin\alpha$이다. 즉 처음의 $\sin(\theta+\alpha)$ 위상자(빨간색)를 sin축 성분과 cos축 성분의 두 위상자(파란색)로 분해할 수 있고 이들의 길이는 각각 $\cos\alpha$, $\sin\alpha$이므로, $\sin(\theta+\alpha)=\cos\alpha\sin\theta+\sin\alpha\cos\theta$가 성립한다.




그림과 같이 기준각이 $\alpha$인 위상 평면에서 $\sin(\alpha+\beta)$, $\sin(\alpha-\beta)$는 길이가 1인 $\sin\alpha$축 위상자가 각각 $\beta$, $-\beta$만큼 회전한 것이다. 따라서 이들을 합성한 보라색 위상자는 $\sin\alpha$축으로 길이가 $2\cos\beta$인 위상자이므로 $\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta$이고 증명이 완료되었다.

이와 비슷한 방법으로 (1), (2), (5), (6), (7)의 공식들을 모두 증명할 수 있다.

**(12) 부정적분의 기본 성질**

① $\int kf(x)dx = k\int f(x)dx \quad (k \in \mathbb{R})$

② $\int \{f(x) \pm g(x)\}dx = \int f(x)dx \pm \int g(x)dx$ (복부호동순)

**(13) 치환적분법**

① $g(x) = t$일 때, $\int f(g(x))g'(x)dx = \int f(t)dt$

② $\int f(x)dx = F(x) + C$일 때, $\int f(ax+b)dx = \frac{1}{a}F(ax+b) + C$

③ $\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b| + C$

④ $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C$

**(14) 바이어슈트라스 치환 (Weierstrass Substitution)**

- 바이어슈트라스 치환은 삼각함수의 유리 적분을 유리식의 적분으로 바꿔주는 치환법이다.

$$\tan\frac{x}{2} = t$$

$$\sin x = \frac{2t}{1+t^2}, \ \cos x = \frac{1-t^2}{1+t^2}, \ \tan x = \frac{2t}{1-t^2}$$

**(15) 오일러 치환 (Euler's Substitution)**

- 오일러 치환은 유리 이변수 함수 $R$에 대하여 다음과 같은 부정적분을 계산하기 위한 치환법이다.

$$\int R(x, \ \sqrt{ax^2+bx+c})dx$$

[1] 제 1종 오일러 치환

$a > 0$일 때,

$$\sqrt{ax^2+bx+c} = \pm x\sqrt{a} + t,$$

$$x = \frac{c-t^2}{\pm 2t\sqrt{a} - b}$$

와 같이 치환한다.

[2] 제 2종 오일러 치환
$c>0$일 때,

$$\sqrt{ax^2+bx+c}=xt\pm\sqrt{c},$$

$$x=\frac{\pm 2t\sqrt{c}-b}{a-t^2}$$

와 같이 치환한다.

[3] 제 3종 오일러 치환
방정식 $ax^2+bx+c=0$이 두 실근 $\alpha$, $\beta$를 가질 때,

$$\sqrt{ax^2+bx+c}=\sqrt{a(x-\alpha)(x-\beta)}=(x-\alpha)t,$$

$$x=\frac{\alpha\beta-\alpha t^2}{a-t^2}$$

와 같이 치환한다.

### (16) 부정적분 공식 ($C\in\mathbb{R}$)

1. $\int dx=x+C$
2. $\int a\,dx=ax+C$ $(a\in\mathbb{R})$
3. $\int x^n dx=\frac{1}{n+1}x^{n+1}+C$ $(-1\neq n\in\mathbb{R})$
4. $\int \frac{1}{x}dx=\ln|x|+C$
5. $\int e^x dx=e^x+C$
6. $\int a^x dx=\frac{a^x}{\ln a}+C$ $\quad(0<a\neq 1)$
7. $\int \ln x\,dx=x\ln x-x+C$
8. $\int \sin x\,dx=-\cos x+C$
9. $\int \cos x\,dx=\sin x+C$
10. $\int \tan x\,dx=\ln|\sec x|+C$
11. $\int \csc x\,dx=\ln|\csc x-\cot x|+C$
12. $\int \sec x\,dx=\ln|\sec x+\tan x|+C$

13. $\int \cot x dx = \ln|\sin x| + C$

14. $\int \sec^2 x dx = \tan x + C$

15. $\int \csc^2 x dx = -\cot x + C$

16. $\int \sec x \tan x dx = \sec x + C$

17. $\int \csc x \cot x dx = -\csc x + C$

18. $\int \frac{1}{(x+a)(x+b)} dx = \frac{1}{b-a} \ln\left|\frac{x+a}{x+b}\right| + C$

19. $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

20. $\int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$

21. $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$

22. $\int \sin^{-1} dx = x \sin^{-1} x + \sqrt{1 - x^2} + C$

23. $\int \cos^{-1} x dx = x \cos^{-1} x - \sqrt{1 - x^2} + C$

24. $\int \tan^{-1} x dx = x \tan^{-1} x - \frac{1}{2} \ln(1 + x^2) + C$

### (17) 헤비사이드 법 (부분분수 분해)

$\frac{g(x)}{f(x)} = \frac{g(x)}{(x-a_1)(x-a_2)\cdots(x-a_n)} = \sum_{i=1}^{n} \frac{b_i}{x-a_i} = \frac{b_1}{x-a_1} + \frac{b_2}{x-a_2} + \cdots + \frac{b_n}{x-a_n}$ 일 때

$\frac{f(x)}{(x-a_i)} = h_i(x)$라 하면 $b_i = \frac{g(a_i)}{h_i(a_i)}$가 성립한다. $(1 \le i \le n)$

### (18) 부분적분법

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

### (19) 삼각함수의 거듭제곱의 부정적분 공식 (Reduction Formula)

$n \in \mathbb{N} - \{1\}$,

$$\int \sin^n x dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x dx$$

$$\int \cos^n x dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x dx$$

$$\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx$$

### (20) 이상적분 (Improper Integration)

- 이상적분은 적분구간의 끝 값이 특정 실수값 또는 $\pm\infty$로 접근할 때의 정적분이다. (양끝값이 모두 극한으로써 작용할 수도 있다.)

- 즉,

$$\lim_{b\to\infty}\int_a^b f(x)dx,\quad \lim_{a\to-\infty}\int_a^b f(x)dx$$

$$\lim_{c\to b-}\int_a^c f(x)dx,\quad \lim_{c\to a+}\int_c^b f(x)dx$$

등과 같은 정적분이다. (Apostol, T (1967), Calculus, Vol. 1 (2nd ed.), Jon Wiley & Sons.) 또한 기호의 남용(abuse of notation)에 의해 적분구간에 $\pm\infty$ 등을 포함하여 쓰기도 한다.

- 예를 들어, $\int_1^\infty \frac{1}{x^2}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}dx = \lim_{b\to\infty}\left(-\frac{1}{b}+\frac{1}{1}\right) = 1$이다.

- 또한, $\frac{1}{\sqrt{x}}$의 경우 $x=0$에서 정의되지 않지만 구간 $[0,1]$에서의 이상적분은 정의 가능하다. 즉,

$$\int_0^1 \frac{1}{\sqrt{x}}dx = \lim_{a\to 0+}\int_a^1 \frac{1}{\sqrt{x}}dx = \lim_{a\to 0+}(2-2\sqrt{a}) = 2$$

이다. 이상적분을 극한으로 변환하여 계산할 때 극한이 발산하면 이상적분이 발산한다고 하고, 대표적인 예시로 $\frac{1}{x}$를 0부터 1까지 적분한 이상적분은 발산한다. 이상적분의 정의에 따른 극한이 수렴할 경우 그 수렴값을 이상적분의 값으로 한다.

- 적분구간의 내점에서 함수가 무한대로 발산하는 등 유계가 아닌 구간이 존재하면, 그 점을 기점으로 적분구간을 쪼갠 후 적분을 진행해야 한다. 예를 들어,

$$\int_{-1}^1 \frac{1}{\sqrt[3]{x^2}}dx = \lim_{s\to 0-}\int_{-1}^s \frac{1}{\sqrt[3]{x^2}}dx + \lim_{t\to\infty}\int_t^1 \frac{1}{\sqrt[3]{x^2}}dx$$

$$= \lim_{s\to 0-}3(1-\sqrt[3]{s}) + \lim_{t\to 0+}3(1-\sqrt[3]{t}) = 3+3 = 6.$$

- 그러나 이와 비슷한 $\int_{-1}^1 \frac{1}{x}dx$는 0을 기점으로 한 각각의 적분이 모두 발산하므로 동일한

논리로 계산할 수 없다. (기함수의 적분을 생각하면 직관적으로 0이겠지만 값이 정의되지 않는다.)

### (21) 코시 주요값 (Cauchy Principal Value)

- 오귀스탱 루이 코시가 도입한 코시 주요값은 일반적인 정적분으로 값을 구할 수 없는 일부 이상적분의 값을 구하는 방법 중 하나이다.

[def] 함수 $f : \mathbb{R} \to \mathbb{R}$가 $x = x_0$ 근처에서 발산한다고 하자. 그러면 $a < x_0 < b$에서의 적분

$$\int_a^b f(x)dx$$

는 리만 적분 또는 르벡 적분으로서 그 값이 존재하지 않을 수 있다. 그러나 만약 다음과 같은 극한이 수렴한다면, 이를 코시 주요값으로 정의한다.

$$P\int_a^b f(x)dx := \lim_{\epsilon \to 0+}\left[\int_a^{x_0 - \epsilon} f(x)dx + \int_{x_0 + \epsilon}^b f(x)dx\right]$$

앞서 언급된 $\int_{-1}^1 \frac{1}{x}dx$는 코시 주요값을 적용하면

$$\lim_{a \to 0+}\left(\int_{-1}^{-a} \frac{1}{x}dx + \int_a^1 \frac{1}{x}dx\right) = 0$$

과 같이 계산할 수 있다. 코시 주요값은

$$PV\int f(x)dx,\quad \text{p.v.}\int f(x)dx,\quad \int_L^* f(z)dz$$

등으로 표기한다.

### (22) Cauchy-Schlömilch Transformation

- Cauchy-Schlömilch Substitution 또는 Cauchy-Schlömilch Transformation은

$$u = x - \frac{1}{x}$$

일 때

$$PV\int_{-\infty}^{\infty} F(u)dx = PV\int_{-\infty}^{\infty} F(x)dx$$

임을 이용하는 치환이다. ($F(u)du$가 아니라 $F(u)dx$임에 유의하라.)

pf) $u=x-\frac{1}{x}$, $x^2-ux-1=0$에서

$$x_{\pm}=\frac{u\pm\sqrt{u^2+4}}{2}$$ (복부호동순)

이고

$$\int_{-\infty}^{\infty}F(u)dx=\int_{-\infty}^{0-}F(u)dx_{-}+\int_{0+}^{\infty}F(u)dx_{+}=\int_{-\infty}^{\infty}F(u)(x_{-}'+x_{+}')du$$

이다. 한편

$$x_{\pm}'=\frac{1}{2}\left(1\pm\frac{u}{\sqrt{u^2+4}}\right)$$

이므로 $x_{-}'+x_{+}'=1$이고

$$\int_{-\infty}^{\infty}F(u)dx=\int_{-\infty}^{\infty}F(u)(x_{-}'+x_{+}')du=\int_{-\infty}^{\infty}F(u)du$$

이다. (Glasser, M. L. (1983). A remarkable property of definite integrals. mathematics of computation, 561-563.)

## (23) Glasser's Master Theorem

- (22)번의 $u=x-\frac{1}{x}$를

$$u=x-\sum_{j=1}^{n-1}\frac{a_j}{x-C_j} \quad \cdots \quad [1]$$

로 대체해도

$$PV\int_{-\infty}^{\infty}F(u)dx=PV\int_{-\infty}^{\infty}F(x)dx \quad \cdots \quad [2]$$

가 성립한다는 것이 알려져 있다. (여기서 $\{a_j\}$는 양의 실수열, $C_j$는 양의 실수이다.)

pf) 일반성을 잃지 않고 $C_1 < C_2 < C_3 \cdots$라 하자. 이때 식 [1]은 $x$에 대하여 다음과 같은 $n$차식으로 환원된다.

$$x^n - \left(u - \sum_{j=1}^{n-1} C_j\right)x^{n-1} + \cdots = 0$$

대수학의 기본정리에 의해 이 방정식은 복소 범위에서 (중복을 포함하여) $n$개의 근을 가지고, $n$개의 근 $x_i \in \mathbb{C}$ $(1 \le i \le n)$에 대하여 근과 계수의 관계에 의해

$$x_1 + x_2 + \cdots + x_n = u - \sum_{j=1}^{n-1} C_j$$

가 성립한다. 따라서

$$x_1' + x_2' + \cdots + x_n' = 1$$

이고,

$$I = \left(\int_{-\infty}^{C_1^-} dx_1 + \int_{C_1^+}^{C_2^-} dx_2 + \cdots + \int_{C_{n-1}^+}^{\infty} dx_n\right) F(u)$$

$$= \int_{-\infty}^{\infty} F(u)(x_1' + x_2' + \cdots + x_n')du = \int_{-\infty}^{\infty} F(u)du$$

이다. 이와 같은 논리를 그대로 적용하면 다음과 같은 치환을 적용해도 식 [2]가 성립한다.

$$u = x - \sum_j a_j \cot\left[(x - C_j)^{-1}\right]$$

위 정리를 이용하면 특정한 값으로 수렴하는 아주 복잡한 적분식을 만들어낼 수 있다.

$$\int_{-\infty}^{\infty} \frac{1}{u^2+1} du = \left[\tan^{-1} u\right]_{-\infty}^{\infty} = \pi$$

이므로

$$u = x - \frac{1}{x} - \frac{1}{x+1} - \frac{1}{x+2}$$

를 대입하여 Glasser's Master Theorem을 적용하면

$$\int_{-\infty}^{\infty} \frac{1}{\left(x-\frac{1}{x}-\frac{1}{x+1}-\frac{1}{x+2}\right)^2+1} dx = \pi$$

이고, 식을 전개하여 짝수차항만 추출하면 다음의 정적분을 얻는다.

$$\int_0^{\infty} \frac{x^{14}-15x^{12}+82x^{10}-190x^8+184x^6-60x^4+16x^2}{x^{16}-20x^{14}+156x^{12}-616x^{10}+1388x^8-1792x^6+1152x^4-224x^2+16} dx = \frac{\pi}{2}$$

이와 같이 복잡한 정적분이 주어졌을 때 Glasser's Master Theorem을 이용하기 위해 식을 변형하여 역추적을 하는 것도 하나의 방법이다. (이를 이용하지 않는다면 복소적분과 매우 복잡한 계산과정을 거쳐야할 것이다.)

예제) Glasser's Master Theorem을 이용하여 다음 식이 성립함을 증명하시오.

$$\int_0^{\infty} \frac{x^8-4x^6+9x^4-5x^2+1}{x^{12}-10x^{10}+37x^8-42x^6+26x^4-8x^2+1} dx = \frac{\pi}{2}$$

Glasser, M. L. (1983). A remarkable property of definite integrals. mathematics of computation, 561-563.

### (24) 정적분 테크닉

① 적분 구간을 이용한 치환 ($x \mapsto a+b-x$)

$$\int_a^b f(x)dx = \int_b^a f(a+b-x)(-dx) = \int_a^b f(a+b-x)dx$$

- $f(x)+f(a+b-x)$의 정적분이 쉽게 계산되는 경우

$$\int_a^b f(x)dx = \int_a^b \frac{f(x)+f(a+b-x)}{2} dx$$

를 이용하여 정적분을 계산할 수 있다.

② 대칭 치환 ($x \mapsto -x$)

$$\int_{-c}^{c} f(x)dx = \int_{c}^{-c} f(-x)(-dx) = \int_{-c}^{c} f(-x)dx$$

- 이는 ①에서 $a+b=0$인 특수한 경우이지만, 마찬가지로 자주 등장한다.

$$\int_{-c}^{c} f(x)dx = \int_{-c}^{c} \frac{f(x)+f(-x)}{2}dx$$

**(25) 파인만 적분 테크닉 (The Feynman Integration Technique)**

- 다음과 같은 적분을 생각하자.

$$I = \int e^{ax}dx = \frac{1}{a}e^{ax} + C$$

여기서 $x$는 적분변수이고, $a$는 상수이다. 따라서 $x$에 대해 미분 또는 적분을 할 때 $a$는 관여하지 않는다. 하지만 관점을 조금 바꿔 보면 피적분함수인 $e^{ax}$를

$$f(x, a) = e^{ax}$$

인 이변수함수로 볼 수도 있을 것이다. 이 관점에 따라 위 적분의 양변을 $a$에 대하여 미분(즉, 편미분) 해보면,

$$\frac{d}{da}\int e^{ax}dx = \frac{\partial}{\partial a}\left[\frac{1}{a}e^{ax} + C\right] = -\frac{1}{a^2}e^{ax} + \frac{x}{a}e^{ax} = \frac{(ax-1)e^{ax}}{a^2} \quad \cdots \quad [3]$$

이다. 한편

$$\frac{d}{da}\int e^{ax}dx = \int\left[\frac{\partial}{\partial a}e^{ax}\right]dx = \int xe^{ax}dx$$

이고 실제로 이는 [3]의 결과와 일치한다. 리처드 파인만에 의해 유명해진 이 적분법은 다음과 같은 라이프니츠 적분 규칙(Leibniz Integral Rule)의 특수한 경우이다.

$$\frac{\partial}{\partial z}\int_{a(z)}^{b(z)} f(x, z)dx = \int_{a(z)}^{b(z)} \frac{\partial f}{\partial z}dx + f(b(z), z)\frac{\partial b}{\partial z} - f(a(z), z)\frac{\partial a}{\partial z}$$

즉, 함수 $f(x, \alpha)$가 $\alpha$에 대해 미분가능하고 $\frac{\partial f}{\partial \alpha}$가 연속함수이면 다음이 성립한다.

$$\frac{d}{d\alpha}\int_{a}^{b} f(x, \alpha)dx = \int_{a}^{b} \frac{\partial}{\partial \alpha}f(x, \alpha)dx$$

이 테크닉을 이용하면 다음과 같은 특이한 정적분의 값을 구할 수 있다.

$$\phi(a)=\int_0^{\pi}\ln(1-2\alpha\cos x+\alpha^2)dx=2\pi\ln|\alpha|\ \ (|\alpha|>1)$$

이 테크닉을 사용하기 위해서는 피적분함수를 강제적으로 이변수함수로 바꿔야 하므로, 특정 숫자를 $\alpha$로 바꾸어 매개변수를 강제로 추가하거나 $e^{-\alpha x}$ 등을 곱해 이변수함수로 만들어준 후 $\alpha=0$일 때의 값을 구하는 식으로 처리한다.

또한 피적분식의 결과를 $\alpha$에 대하여 미분한 값이 $\alpha$에 대한 함수로 나오므로 이는 $\alpha$에 대한 미분방정식을 푸는 것으로 이어진다. (복소적분 외에) 파인만 테크닉을 이용해야 하는 문제들은 그 미분방정식의 해법이 어렵지 않은 것들만 수록하였다.


[1] Feynman, R. P. "A Different Set of Tools." In 'Surely You're Joking, Mr. Feynman!': Adventures of a Curious Character. New York: W. W. Norton, 1997.

[2] Hijab, O. Introduction to Calculus and Classical Analysis. New York: Springer-Verlag, p. 189, 1997.

[3] Woods, F. S. "Differentiation of a Definite Integral." §60 in Advanced Calculus: A Course Arranged with Special Reference to the Needs of Students of Applied Mathematics. Boston, MA: Ginn, pp. 141-144, 1926.


### (26) 역함수의 적분 공식

- 실함수 $f$와 그 역함수 $f^{-1}$, $F(x)=\int f(x)dx$에 대하여 다음이 성립한다. (양변을 미분한 후 chain rule을 적용하면 쉽게 증명된다.)

$$\int f^{-1}(x)dx=x\cdot f^{-1}(x)-F\left(f^{-1}(x)\right)+C$$

※ 적분구간에 $\pm\infty$ 또는 그 지점에서 함숫값이 정의되지 않는 값이 포함된 경우, 즉 이상적분의 경우 비록 교과 외 과정이긴 하나 (20)의 설명만으로도 충분히 풀 수 있을 것이라 생각하여 따로 표시하지 않았습니다. 간혹 $\exp x$라는 표기가 등장하는데, 이는 $e^x$와 같으며 $x$ 자리에 복잡한 분수식이 들어갔을 때 식이 너무 작아져 가독성이 떨어지는 것을 방지하기 위한 표기법입니다.

※※ 쌍곡선함수($\sinh x$, $\cosh x$, $\tanh x$)와 그 역함수는 등장하지 않으며, 역쌍곡선함수의 경우 $\tanh^{-1}x$라는 표기 대신 $\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|$ 등으로 나타내었습니다. 적분 결과 또는 피적분함수에 역삼각함수가 등장하는 경우 고등학교 교육과정을 벗어나는 것이기는 하나, (16)의 19, 21번 공식을 사용하는 수준에서 모두 해결 가능합니다.

※※※ 위에서 언급된 테크닉과 고등학교 미적분 수준을 벗어나는 문제는 없습니다. 즉 모든 문제를 고등학교 미적분 교과 내용과 위에서 언급된 테크닉을 이용하여 해결할 수 있고, 유수 정리(Residue Theorem)와 경로 적분(Contour Integration) 등 복소해석학의 정리가 사용될 일은 없습니다.

※※※※ 해설을 보아도 알 수 있겠지만, 모든 문제는 정확한 수치를 구할 수 있고, 부정적분과 달리 그 값이 단 하나로 특정됩니다.

[1~200] 다음 정적분의 값을 구하시오.

1. $\int_0^{\pi} \frac{x\sin x}{3+\sin^2 x}dx$

2. $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}dx$

3. $\int_0^{\frac{\pi}{2}} \frac{\cos^{\sqrt{5}} x}{\sin^{\sqrt{5}} x+\cos^{\sqrt{5}} x}dx$

4. $\int_0^{\frac{\pi}{4}} \frac{\sin x}{\sin x+\cos x}dx$

5. $\int_0^{8} \frac{x^3-2x+1}{\sqrt[3]{x}}dx$

6. $\int_{-3}^{3} \frac{2x^2}{2^x+1}dx$

7. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{e^{1/x}+1}dx$

8. $\int_0^{\frac{\pi}{2}} \sin 3x\cos 4x dx$

9. $\int_0^{\frac{\pi}{2}} \sqrt[3]{\tan x}\, dx$

10. $\int_0^{\frac{\pi}{4}} \sqrt[3]{\tan x}\, dx$

11. $\int_0^{1} \frac{1}{x^x}dx$ (급수의 형태로 나타내어도 무방함.)

12. $\int_0^{1} \frac{1}{1+\left(1-\frac{1}{x}\right)^{2022}}dx$

13. $\int_{-\infty}^{\infty} e^{-x^2}dx$

14. $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx$

15. $\int_0^\infty 4\pi r^2 |\psi(r)|^2 dr \ \ (\psi(r) = \frac{1}{\sqrt{\pi a_0^3}} e^{-\frac{r}{a_0}})$

16. $\int_{\frac{a_0}{2}}^\infty 4\pi r^2 |\psi(r)|^2 dr \ \ (\psi(r) = \frac{1}{\sqrt{\pi a_0^3}} e^{-\frac{r}{a_0}})$

17. $\int_0^{\frac{\pi}{2}} \sin^{10} x dx$

18. $\int_0^1 \frac{x^{2022} - 1}{\ln x} dx$

19. $\int_0^1 \frac{\ln x}{x - 1} dx$

20. $\int_{-\infty}^\infty \frac{\sin x}{x} dx$

21. $\int_{-\infty}^\infty \frac{\sin^2 x}{x^2} dx$

22. $\int_0^1 \sqrt[3]{2x^3 - 3x^2 - x + 1}\, dx$

23. $\int_0^1 \frac{\ln(x+1)}{x^2 + 1} dx$

24. $\int_0^2 \frac{\ln(x+1)}{x^2 - x + 1} dx$

25. $\int_0^1 \frac{\ln(x^2+1)}{x + 1} dx$

26. $\int_0^{\frac{\pi}{2}} \ln(\cos x) dx$

27. $\int_0^1 \frac{\ln x}{x + 1} dx$

28. $\int_0^{\frac{\pi}{2}} \cos^{2022} x \cos 2022x dx$

29. $\int_0^1 (\ln x)^{2022} dx$

30. $\int_1^e \frac{\ln x}{\sqrt{x}} dx$

31. $\int_0^{\frac{\pi}{2}} \ln(2 + \tan^2 x) dx$

32. $\int_0^\infty \frac{x - \sin x}{x^3(x^2+4)} dx$

33. $\int_0^\pi \frac{1 - \sin x}{1 + \sin x} dx$

34. $\int_0^{\frac{3}{5}} \frac{e^x(2-x^2)}{(1-x)\sqrt{1-x^2}}dx$

35. $\int_0^{\frac{\pi}{2}} \sin 2022x \cdot \sin^{2020}x dx$

36. $\int_1^{\sqrt{2}} \frac{x^4-1}{x^2\sqrt{x^4-x^2+1}}dx$

37. $\int_0^{\frac{\pi}{3}} \frac{\sin 2x}{2+\cos x}dx$

38. $\int_0^{\frac{\pi}{4}} \tan^9 x dx$

39. $\int_0^1 \frac{x-1}{(1+x^3)\ln x}dx$

40. $\int_0^{\frac{\pi}{2}} \frac{(1+\sec^2 t)\sqrt{\sec t}}{(1+\sec t)^2-2}dt$

41. $\int_0^{\frac{\pi}{2}} \frac{\cos x}{2-\sin 2x}dx$

42. $\int_0^{\infty} (\ln(e^x+1)-x)dx$

43. $\int_0^{\infty} \frac{x}{e^x+1}dx$

44. $\int_0^{\frac{\pi}{2}} \frac{\ln(\tan x)}{1-\tan x+\tan^2 x}dx$

45. $\int_{-\infty}^{\infty} \frac{\cos\left(x-\frac{1}{x}\right)}{1+x^2}dx$

46. $\int_0^{\infty} \frac{\cos x}{x^2+1}dx$

47. $\int_0^{\infty} \frac{x\sin x}{x^2+1}dx$

48. $\int_0^{\infty} \frac{\ln(1+x)}{x(x^2+1)}dx$

49. $\int_0^{\frac{\pi}{2}} \frac{\sin(u+\sqrt{\pi}\tan u)}{\sin u}du$

50. $\int_0^{\pi} \sin^2\left(x-\sqrt{\pi^2-x^2}\right)dx$

51. $\int_0^{\infty} e^{-x^2}\cos 2x dx$

52. $\int_0^{\frac{\pi}{2}} \cos(\tan x)dx$

53. $\int_0^{\frac{\pi}{2}} \frac{x}{\tan x} dx$

54. $\int_0^{\pi} \ln(1-2e\cos x+e^2) dx$

55. $\int_0^{\pi} \ln(1+\sin^2 t) dt$

56. $\int_0^{4} \frac{\ln x}{\sqrt{4x-x^2}} dx$

57. $\int_0^{\infty} \frac{1}{(1+x^2)(1+x^{\pi})} dx$

58. $\int_0^{\infty} \frac{1}{(1+x)(\pi^2+(\ln x)^2)} dx$

59. $\int_0^{\infty} \frac{x-1}{\sqrt{2^x-1}\ln(2^x-1)} dx$

60. $\int_0^{3\pi} \frac{1}{\sin^4 x+\cos^4 x} dx$

61. $\int_0^{\pi} \cos^4 x dx$

62. $\int_0^{\frac{\pi}{2}} \sin^6 x\cos^3 x dx$

63. $\int_0^{\frac{\pi}{4}} \sin^3 x\sec^2 x dx$

64. $\int_1^{3} \frac{1}{x^2\sqrt{x^2+4}} dx$

65. $\int_{\sqrt{3}-1}^{2\sqrt{2}-1} \frac{1}{(x+1)\sqrt{x^2+2x+2}} dx$

66. $\int_0^{\tan^{-1}(\sqrt{6})} \frac{1}{1+\cos^2 x} dx$

67. $\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x} dx$

68. $\int_{\ln(e-1)}^{\ln(e^3-1)} \frac{e^x\ln(e^x+1)}{e^x+1} dx$

69. $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{\sin x} dx$

70. $\int_0^{\frac{\pi}{4}} \frac{1}{1-3\cos^2 x} dx$

71. $\int_{-1}^{0} \frac{x^3-x-2}{x^3-x^2+x-1} dx$

72. $\int_{3}^{4} \frac{x+4}{x^3+3x^2-10x}\,dx$

73. $\int_{5}^{6} \frac{7x^3-13x^2-24x+24}{x^4-3x^3-10x^2+24x}\,dx$

74. $\int_{1}^{2} \frac{1}{x^2\sqrt{2x-x^2}}\,dx$

75. $\int_{1}^{e} \frac{x^4+81}{x(x^2+9)^2}\,dx$

76. $\int_{0}^{3} \sqrt{\frac{4-x}{x}}\,dx$

77. $\int_{0}^{\frac{1}{\sqrt[3]{2}}} \sqrt{\frac{x}{1-x^3}}\,dx$

78. $\int_{0}^{\frac{3}{4}} \sqrt{x}\,\sqrt{1-x}\,dx$

79. $\int_{0}^{\tan^{-1}(\sqrt[4]{3})} \frac{\sin x\cos x}{\sin^4 x+\cos^4 x}\,dx$

80. $\int_{0}^{\frac{\pi}{4}} \tan^4 x\sec^4 x\,dx$

81. $\int_{2}^{4} \frac{x\sqrt{x}-1}{x-\sqrt{x}}\,dx$

82. $\int_{0}^{\ln 3} \frac{1}{\sqrt{1+e^x}}\,dx$

83. $\int_{1}^{2} \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$

84. $\int_{0}^{\frac{\pi}{3}} 13e^{2x}\cos 3x\,dx$

85. $\int_{0}^{\pi} e^{-x}\sin^2 2x\,dx$

86. $\int_{0}^{\frac{\pi}{4}} \sqrt{\tan x}\,dx$

87. $\int_{0}^{\frac{\pi}{4}} \sqrt{\tan x}\,\sqrt{1-\tan x}\,dx$

88. $\int_{1}^{\infty} \frac{(\ln x)^{627}}{x^{2022}}\,dx$

89. $\int_{0}^{\sqrt{6}} \cos^2\left(\tan^{-1}\left(\sin\left(\cot^{-1}x\right)\right)\right)dx$

90. $\int_{0}^{\frac{\pi}{6}} \frac{\sec^2 x}{(\sec x+\tan x)^{5/2}}\,dx$

91. $\displaystyle\int_{1}^{2}\frac{1}{x\sqrt{x^2+4x-4}}\,dx$

92. $\displaystyle\int_{1}^{\frac{25}{73}}\frac{1}{x\sqrt{-x^2+x+2}}\,dx$

93. $\displaystyle\int_{1}^{2}\frac{x^2}{\sqrt{-x^2+3x-2}}\,dx$

94. $\displaystyle\int_{0}^{\frac{\pi}{4}}\frac{\sin^3(\theta/2)}{\cos(\theta/2)\cdot\sqrt{\cos^3\theta+\cos^2\theta+\cos\theta}}\,d\theta$

95. $\displaystyle\int_{0}^{\frac{\pi}{6}}\frac{\tan^4\theta}{1-\tan^2\theta}\,d\theta$

96. $\displaystyle\int_{-\infty}^{\infty}\frac{x^2}{x^4+1}\,dx$

97. $\displaystyle\int_{0}^{\infty}\exp\left(-x^2-\frac{1}{x^2}\right)dx$

98. $\displaystyle\int_{0}^{1}\frac{1-x^{99}}{(1+x)(1+x^{100})}\,dx$

99. $\displaystyle\int_{-\infty}^{\infty}\exp\left(-\frac{(x^2-13x-1)^2}{611x^2}\right)dx$

100. $\displaystyle\int_{0}^{\frac{\pi}{2}}\sqrt[n]{\tan x}\,dx \quad (2\le n\in\mathbb{N})$

[해설]

1. $I=\int_0^{\pi}\frac{x\sin x}{3+\sin^2 x}dx=\int_0^{\pi}\frac{(\pi-x)\sin x}{3+\sin^2 x}dx\ \ (x\ \mapsto\ \pi-x)$

$2I=\int_0^{\pi}\left(\frac{x\sin x}{3+\sin^2 x}+\frac{(\pi-x)\sin x}{3+\sin^2 x}\right)dx=\int_0^{\pi}\frac{\pi\sin x}{3+\sin^2 x}dx=\int_0^{\pi}\frac{\pi\sin x}{4-\cos^2 x}dx$

$=\int_1^{-1}\frac{\pi}{u^2-4}du\ \ (u=\cos x,\ du=-\sin x dx)$

$=\pi\left[\frac{1}{4}\ln\left|\frac{u-2}{u+2}\right|\right]_1^{-1}=\frac{\pi}{2}\ln 3$

$\therefore I=\frac{\pi}{4}\ln 3$ ■

2. $I=\int_0^{\frac{\pi}{2}}\frac{\sin x}{\sin x+\cos x}dx$

$\frac{\pi}{2}-2I=\int_0^{\frac{\pi}{2}}dx-\int_0^{\frac{\pi}{2}}\frac{2\sin x}{\sin x+\cos x}dx=\int_0^{\frac{\pi}{2}}\frac{\cos x-\sin x}{\sin x+\cos x}dx=[\ln|\sin x+\cos x|]_0^{\frac{\pi}{2}}=0$

$\therefore I=\frac{\pi}{4}$ ■

3. $I=\int_0^{\frac{\pi}{2}}\frac{\cos^{\sqrt{5}}x}{\sin^{\sqrt{5}}x+\cos^{\sqrt{5}}x}dx=\int_0^{\frac{\pi}{2}}\frac{\sin^{\sqrt{5}}x}{\sin^{\sqrt{5}}x+\cos^{\sqrt{5}}x}dx\ \ (x\ \mapsto\ \frac{\pi}{2}-x)$

$2I=\int_0^{\frac{\pi}{2}}\left(\frac{\cos^{\sqrt{5}}x}{\sin^{\sqrt{5}}x+\cos^{\sqrt{5}}x}+\frac{\sin^{\sqrt{5}}x}{\sin^{\sqrt{5}}x+\cos^{\sqrt{5}}x}\right)dx=\frac{\pi}{2}$

$\therefore I=\frac{\pi}{4}$ ■

4. $\int_0^{\frac{\pi}{4}}\frac{\sin x}{\sin x+\cos x}dx=\int_0^{\frac{\pi}{4}}\frac{\sin\left(\frac{\pi}{4}-x\right)}{\sin\left(\frac{\pi}{4}-x\right)+\cos\left(\frac{\pi}{4}-x\right)}dx\ \ (x\ \mapsto\ \frac{\pi}{4}-x)$

$$=\frac{1}{2}\int_0^{\frac{\pi}{4}}\frac{\cos x-\sin x}{\cos x}dx=\frac{1}{2}\int_0^{\frac{\pi}{4}}(1-\tan x)dx=\frac{1}{2}\left[x+\ln|\cos x|\right]_0^{\frac{\pi}{4}}=\frac{\pi}{8}-\frac{1}{4}\ln 2\ \blacksquare$$

5. $t=\sqrt[3]{x}$ , $dx=3t^2dt$

$$\int_0^8\frac{x^3-2x+1}{\sqrt[3]{x}}dx=\int_0^2 3t(t^9-2t^3+1)dt=\left[\frac{3}{11}t^{11}-\frac{6}{5}x^5+\frac{3}{2}t^2\right]_0^2$$

$$=\left[\frac{3}{11}x^{\frac{11}{3}}-\frac{6}{5}x^{\frac{5}{3}}+\frac{3}{2}x^{\frac{2}{3}}\right]_0^8=\frac{28938}{55}\ \blacksquare$$

6. $I=\int_{-3}^{3}\frac{2x^2}{2^x+1}dx=\int_{-3}^{3}\frac{2x^2}{2^{-x}+1}dx=\int_{-3}^{3}\frac{2^x\cdot 2x^2}{2^x+1}dx\ \ (x\mapsto -x)$

$$2I=\int_{-3}^{3}\left(\frac{2x^2}{2^x+1}+\frac{2^x\cdot 2x^2}{2^x+1}\right)dx=\int_{-3}^{3}2x^2dx=\left[\frac{2}{3}x^3\right]_{-3}^{3}=36$$

$$\therefore I=\int_{-3}^{3}\frac{2x^2}{2^x+1}dx=18\ \blacksquare$$

7. $I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos x}{e^{1/x}+1}dx=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos x}{e^{-1/x}+1}dx=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{e^{1/x}\cdot\cos x}{e^{1/x}+1}dx\ \ (x\mapsto -x)$

$$2I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{\cos x}{e^{1/x}+1}+\frac{e^{1/x}\cdot\cos x}{e^{1/x}+1}\right)dx=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos x\,dx=2$$

$$\therefore I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos x}{e^{1/x}+1}dx=1\ \blacksquare$$

8. $\int_0^{\frac{\pi}{2}}\sin 3x\cos 4x\,dx=\frac{1}{2}\int_0^{\frac{\pi}{2}}(\sin 7x-\sin x)dx=\left[\frac{1}{2}\cos x-\frac{1}{14}\cos 7x\right]_0^{\frac{\pi}{2}}=-\frac{3}{7}\ \blacksquare$

9. [1] $t^3=\tan x$, $x=\tan^{-1}(t^3)$, $dx=\frac{3t^2}{1+t^6}dt$

[2] $a=t^2$, $da=2tdt$, $a=\tan^{2/3}x$

$$\int_0^{\frac{\pi}{2}}\sqrt[3]{\tan x}\,dx=\int_0^{\infty}\frac{3t^3}{1+t^6}dt\ \ \cdots\ \ [1]$$

$$= \frac{3}{2}\int_0^{\infty} \frac{a}{1+a^3}da \ \cdots \ [2]$$

$$= \frac{3}{2}\int_0^{\infty}\left(-\frac{1}{3(a+1)}+\frac{a+1}{3(a^2-a+1)}\right)da$$

$$= \frac{1}{2}\int_0^{\infty}\left(\frac{2a-1}{2(a^2-a+1)}+\frac{3}{2(a^2-a+1)}-\frac{1}{a+1}\right)da$$

$$= \left[\frac{1}{4}\ln\frac{a^2-a+1}{(a+1)^2}+\frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(a-\frac{1}{2}\right)\right)\right]_0^{\infty}$$

$$= \left[\frac{1}{4}\ln\frac{\tan^{4/3}x-\tan^{2/3}x+1}{\left(\tan^{2/3}x+1\right)^2}+\frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(\tan^{2/3}x-\frac{1}{2}\right)\right)\right]_0^{\frac{\pi}{2}}$$

$$= \lim_{x\to\frac{\pi}{2}}\left\{\frac{1}{4}\ln\frac{\tan^{4/3}x-\tan^{2/3}x+1}{\left(\tan^{2/3}x+1\right)^2}+\frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(\tan^{2/3}x-\frac{1}{2}\right)\right)\right\}-\frac{\sqrt{3}}{2}\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$$

$$= \lim_{t\to\infty}\left\{\frac{1}{4}\ln\frac{t^{4/3}-t^{2/3}+1}{\left(t^{2/3}+1\right)^2}+\frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(t^{2/3}-\frac{1}{2}\right)\right)\right\}+\frac{\sqrt{3}}{12}\pi$$

$$= \frac{1}{4}\ln 1+\frac{\sqrt{3}}{4}\pi+\frac{\sqrt{3}}{12}\pi=\frac{\pi}{\sqrt{3}} \ \blacksquare$$

10. [1] $t^3=\tan x$, $x=\tan^{-1}(t^3)$, $dx=\frac{3t^2}{1+t^6}dt$

[2] $a=t^2$, $da=2tdt$, $a=\tan^{2/3}x$

$$\int_0^{\frac{\pi}{4}}\sqrt[3]{\tan x}\,dx=\int_0^1\frac{3t^3}{1+t^6}dt \ \cdots \ [1]$$

$$= \frac{3}{2}\int_0^1\frac{a}{1+a^3}da \ \cdots \ [2]$$

$$= \frac{3}{2}\int_0^1\left(-\frac{1}{3(a+1)}+\frac{a+1}{3(a^2-a+1)}\right)da$$

$$= \frac{1}{2}\int_0^1 \left( \frac{2a-1}{2(a^2-a+1)} + \frac{3}{2(a^2-a+1)} - \frac{1}{a+1} \right) da$$

$$= \left[ \frac{1}{4}\ln\frac{a^2-a+1}{(a+1)^2} + \frac{\sqrt{3}}{2}\tan^{-1}\left( \frac{2}{\sqrt{3}}\left( a - \frac{1}{2} \right) \right) \right]_0^1$$

$$= \left[ \frac{1}{4}\ln\frac{\tan^{4/3}x - \tan^{2/3}x + 1}{\left(\tan^{2/3}x + 1\right)^2} + \frac{\sqrt{3}}{2}\tan^{-1}\left( \frac{2}{\sqrt{3}}\left( \tan^{2/3}x - \frac{1}{2} \right) \right) \right]_0^{\frac{\pi}{4}}$$

$$= \left( -\frac{1}{2}\ln 2 + \frac{\sqrt{3}}{12}\pi \right) - \left( 0 - \frac{\sqrt{3}}{12}\pi \right) = \frac{\sqrt{3}}{6}\pi - \frac{1}{2}\ln 2 \ ■$$

11. $\int_0^1 \frac{1}{x^x}dx = \int_0^1 e^{-x\ln x}dx = \int_0^1 \sum_{n=0}^{\infty}\frac{(-x\ln x)^n}{n!}dx = \sum_{n=0}^{\infty}\int_0^1 \frac{(-x\ln x)^n}{n!}dx$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^1 x^n(\ln x)^n dx$$

한편

$$\int_0^1 x^n(\ln x)^n dx = \left[ \frac{1}{n+1}x^{n+1}(\ln x)^n \right]_0^1 - \int_0^1 \frac{1}{n+1}x^{n+1} \cdot \frac{n(\ln x)^{n-1}}{x}dx$$

$$= -\frac{n}{n+1}\int_0^1 x^n(\ln x)^{n-1}dx = (-1)^2\frac{n(n-1)}{(n+1)^2}\int_0^1 x^n(\ln x)^{n-2}dx$$

$$= (-1)^3\frac{n(n-1)(n-2)}{(n+1)^3}\int_0^1 x^n(\ln x)^{n-3}dx = \cdots$$

$$= (-1)^n\frac{n!}{(n+1)^n}\int_0^1 x^n(\ln x)^{n-n}dx = (-1)^n\frac{n!}{(n+1)^{n+1}}$$

이므로

$$\therefore \int_0^1 \frac{1}{x^x}dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^1 x^n(\ln x)^n dx = \sum_{n=0}^{\infty}(n+1)^{-(n+1)} = \sum_{n=1}^{\infty}\frac{1}{n^n} \ ■$$

12. $I = \int_0^1 \frac{1}{1+\left(1-\frac{1}{x}\right)^{2022}}dx = \int_0^1 \frac{1}{1+\left(\frac{x-1}{x}\right)^{2022}}dx = \int_0^1 \frac{x^{2022}}{x^{2022}+(x-1)^{2022}}dx$

$$= \int_0^1 \frac{(1-x)^{2022}}{x^{2022}+(x-1)^{2022}}\,dx \ \ (x \mapsto 1-x)$$

$$= \frac{1}{2}\int_0^1 \left(\frac{x^{2022}}{x^{2022}+(1-x)^{2022}} + \frac{(1-x)^{2022}}{x^{2022}+(1-x)^{2022}}\right)dx = \frac{1}{2} \ ■$$

13. $I = \int_{-\infty}^{\infty} e^{-x^2}dx = 2\int_0^{\infty} e^{-x^2}dx$ (Gaussian Integral, 가우스 적분)

$$I^2 = 4\int_0^{\infty} e^{-x^2}dx \cdot \int_0^{\infty} e^{-y^2}dy = 4\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}dxdy$$

$$= 4\int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-r^2}rdrd\theta \ \ \left(r = x^2+y^2,\ \theta = \tan^{-1}\left(\frac{y}{x}\right),\ dxdy = rdrd\theta\right)$$

$$= 4\int_0^{\frac{\pi}{2}} d\theta \cdot \int_0^{\infty} re^{-r^2}dr = 2\pi\left[-\frac{1}{2}e^{-r^2}\right]_0^{\infty} = \pi$$

$$\therefore I = \int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi} \ ■$$

14. $z = \frac{x-\mu}{\sigma},\ dz = \frac{dx}{\sigma}$ ($N(\mu, \sigma^2)$을 따르는 정규분포의 확률밀도함수)

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}dz$$

$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-y^2}dy \ \ (z = \sqrt{2}\,y,\ dz = \sqrt{2}\,dy)$$

$$= 1 \ ■$$

15. $\int_0^{\infty} 4\pi r^2 |\psi(r)|^2 dr = \frac{4}{a_0^3}\int_0^{\infty} r^2 e^{-\frac{2r}{a_0}}dr = \frac{4}{a_0^3}\left\{\left[-\frac{a_0}{2}r^2 e^{-\frac{2r}{a_0}}\right]_0^{\infty} + \frac{a_0}{2}\int_0^{\infty} 2re^{-\frac{2r}{a_0}}dr\right\}$

$$= \frac{4}{a_0^3}\left\{0 + a_0\left[-\frac{a_0}{2}re^{-\frac{2r}{a_0}}\right]_0^{\infty} + \frac{a_0^2}{2}\int_0^{\infty} e^{-\frac{2r}{a_0}}dr\right\}$$

$$= \frac{4}{a_0^3}\left\{\frac{a_0^2}{2}\left[-\frac{a_0}{2}e^{-\frac{2r}{a_0}}\right]_0^{\infty}\right\} = \frac{4}{a_0^3}\left(\frac{a_0^3}{4}\right) = 1 \ ■$$

$\psi(r)$은 거리 $r$에 따른 수소원자의 바닥상태 파동함수이다. 위 적분은 수소원자의 전구간에서 전자가 발견될 확률이 1임을 의미한다.

16. $$\int_{\frac{a_0}{2}}^{\infty} 4\pi r^2 |\psi(r)|^2 dr = \frac{4}{a_0^3}\int_{\frac{a_0}{2}}^{\infty} r^2 e^{-\frac{2r}{a_0}} dr = \frac{4}{a_0^3}\left\{\left[-\frac{a_0}{2}r^2 e^{-\frac{2r}{a_0}}\right]_{\frac{a_0}{2}}^{\infty} + \frac{a_0}{2}\int_{\frac{a_0}{2}}^{\infty} 2re^{-\frac{2r}{a_0}} dr\right\}$$

$$= \frac{4}{a_0^3}\left\{\frac{a_0^3}{8e} + a_0\left[-\frac{a_0}{2}re^{-\frac{2r}{a_0}}\right]_{\frac{a_0}{2}}^{\infty} + \frac{a_0^2}{2}\int_{\frac{a_0}{2}}^{\infty} e^{-\frac{2r}{a_0}} dr\right\}$$

$$= \frac{4}{a_0^3}\left\{\frac{3a_0^3}{8e} + \frac{a_0^2}{2}\left[-\frac{a_0}{2}e^{-\frac{2r}{a_0}}\right]_{\frac{a_0}{2}}^{\infty}\right\} = \frac{4}{a_0^3}\left(\frac{5a_0^3}{8e}\right) = \frac{5}{2e} \simeq 91.97\% \ ■$$

$\psi(r)$은 거리 $r$에 따른 수소원자의 바닥상태 파동함수이다. 위 적분은 수소원자에서 전자가 핵으로부터 $\frac{a_0}{2}$ 보다 멀리서 발견될 확률이 약 91.97%임을 의미한다. ($a_0$는 보어 반지름이다.) [2021 물리인증 1급 기출]

17. $$W_n := \int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \sin^{n-1} x \cdot \sin x\,dx$$

$$= \left[-\cos x \cdot \sin^{n-1} x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (n-1)\sin^{n-2} x \cos x \cdot (-\cos x)\,dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x\,dx = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x (1-\sin^2 x)\,dx$$

$$= (n-1)(W_{n-2} - W_n)$$

$$nW_n = (n-1)W_{n-2}, \quad W_n = \frac{n-1}{n} W_{n-2} \ (n \ge 2)$$

한편

$$W_0 = \int_0^{\frac{\pi}{2}} 1\,dx = \frac{\pi}{2}, \quad W_1 = \int_0^{\frac{\pi}{2}} \sin x\,dx = 1$$

이므로

$$W_n = \frac{n-1}{n} W_{n-2} = \frac{(n-1)(n-3)}{n(n-2)} W_{n-4} = \cdots$$

$$= \begin{cases} \frac{(n-1)(n-3)\cdots 1}{n(n-2)\cdots 2} \cdot W_0 & (n = 2m) \\ \frac{(n-1)(n-3)\cdots 2}{n(n-2)\cdots 1} \cdot W_1 & (n = 2m+1) \end{cases}$$

$$\therefore W_n = \frac{(n-1)!!}{n!!} \cdot \left(\frac{\pi}{2}\right)^{\frac{1+(-1)^n}{2}} \quad (n > 0),\ \ W_0 = \frac{\pi}{2} \ \ \text{(Wallis' Formula)}$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin^{10} x dx = W_{10} = \frac{9 \cdot 7 \cdot 5 \cdot 3 \cdot 1}{10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} = \frac{63\pi}{512} \ ■$$

18. $I(a) := \int_0^1 \frac{x^a - 1}{\ln x} dx$, 파인만 적분 테크닉을 적용하면

$$\frac{d}{da} I(a) = \int_0^1 \frac{\partial}{\partial a}\left(\frac{x^a - 1}{\ln x}\right) dx = \int_0^1 x^a dx = \left[\frac{1}{a+1} x^{a+1}\right]_0^1 = \frac{1}{a+1}$$

$$I(a) = \int \frac{1}{a+1} da = \ln|a+1| + C$$

이때 $I(0) = \int_0^1 0\, dx = 0$이므로 $C = 0$이다.

$$\therefore I(2022) = \int_0^1 \frac{x^{2022} - 1}{\ln x} dx = \ln 2023 \ ■$$

19. $(1+x)^{-1}$의 이항급수를 구하고 양변을 적분하면

$$(1+x)^{-1} = \sum_{n=0}^{\infty} \binom{-1}{n} \cdot x^n = \sum_{n=0}^{\infty} \prod_{i=0}^{n-1} (-1-i) \cdot \frac{x^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \cdot x^n \quad (|x| < 1)$$

$$\int (1+x)^{-1} dx = \ln|1+x| = \int \sum_{n=0}^{\infty} (-1)^n x^n dx + C = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} + C$$

$x = 0$을 대입하면 $0 = 0 + C$에서 $C = 0$이다.

$$\therefore \int_0^1 \frac{\ln x}{x-1}dx = \int_{-1}^0 \frac{\ln(t+1)}{t}dt = \int_{-1}^0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}t^n dt = \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2}t^{n+1}\right]_{-1}^0$$

$$= \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2}0^{n+1}\right] - \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2}(-1)^{n+1}\right] = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2}$$

$= \zeta(2) = \frac{\pi^2}{6}$ ■ ( $\because$ 바젤 문제에 의해 $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ )

20. $I(\alpha) := \int_0^{\infty} \frac{\sin x}{x}e^{-\alpha x}dx$, $\frac{d}{d\alpha}I(\alpha) = \int_0^{\infty} \frac{\partial}{\partial \alpha} \frac{\sin x}{x}e^{-\alpha x}dx = -\int_0^{\infty} e^{-\alpha x}\sin x dx$

$$= -\left[-e^{-\alpha x}\cos x\right]_0^{\infty} + \int_0^{\infty} \alpha e^{-\alpha x}\cos x dx = -1 + \left[\alpha e^{-\alpha x}\sin x\right]_0^{\infty} + \int_0^{\infty} \alpha^2 e^{-\alpha x}\sin x dx$$

$= -1 - \alpha^2 I'(\alpha)$, $\quad I'(\alpha) = -\frac{1}{1+\alpha^2}$

$I(\alpha) = \int I'(\alpha)d\alpha = C - \tan^{-1}\alpha \ (C \in \mathbb{R})$, $\quad 0 = \lim_{\alpha \to \infty} I(\alpha) = C - \frac{\pi}{2}$ 에서 $C = \frac{\pi}{2}$ 이다.

$\therefore \int_{-\infty}^{\infty} \frac{\sin x}{x}dx = 2\int_0^{\infty} \frac{\sin x}{x}dx = 2I(0) = \pi$ ■

21. $\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}dx = \left[-\frac{\sin^2 x}{x}\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{2\sin x \cos x}{x}dx = \int_{-\infty}^{\infty} \frac{\sin 2x}{x}dx$

$= \int_{-\infty}^{\infty} \frac{\sin t}{t}dt \quad (t = 2x,\ dt = 2dx)$

$I(\alpha) := \int_0^{\infty} \frac{\sin x}{x}e^{-\alpha x}dx$, $\frac{d}{d\alpha}I(\alpha) = \int_0^{\infty} \frac{\partial}{\partial \alpha} \frac{\sin x}{x}e^{-\alpha x}dx = -\int_0^{\infty} e^{-\alpha x}\sin x dx$

$$= -\left[-e^{-\alpha x}\cos x\right]_0^{\infty} + \int_0^{\infty} \alpha e^{-\alpha x}\cos x dx = -1 + \left[\alpha e^{-\alpha x}\sin x\right]_0^{\infty} + \int_0^{\infty} \alpha^2 e^{-\alpha x}\sin x dx$$

$= -1 - \alpha^2 I'(\alpha)$, $\quad I'(\alpha) = -\frac{1}{1+\alpha^2}$

$I(\alpha)=\int I'(\alpha)d\alpha = C-\tan^{-1}\alpha \ (C\in\mathbb{R}),\quad 0=\lim_{\alpha\to\infty} I(\alpha) = C-\frac{\pi}{2}$ 에서 $C=\frac{\pi}{2}$ 이다.

$$\therefore \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}dx = \int_{-\infty}^{\infty} \frac{\sin x}{x}dx = 2\int_0^{\infty} \frac{\sin x}{x}dx = 2I(0) = \pi \ ■$$

22. $I=\int_0^1 \sqrt[3]{2x^3-3x^2-x+1}\,dx = \int_0^1 \sqrt[3]{2(1-x)^3-3(1-x)^2-(1-x)+1}\,dx \ (x \mapsto 1-x)$

$$=\int_0^1 -\sqrt[3]{2x^3-3x^2-x+1}\,dx = -I$$

$$\therefore I=\int_0^1 \sqrt[3]{2x^3-3x^2-x+1}\,dx = 0 \ ■$$

23. $x=\tan t,\ dx=\sec^2 t dt,\ \frac{dx}{1+x^2}=dt$

$$I=\int_0^1 \frac{\ln(x+1)}{x^2+1}dx = \int_0^{\frac{\pi}{4}} \ln(1+\tan t)dt = \int_0^{\frac{\pi}{4}} \ln\left(1+\tan\left(\frac{\pi}{4}-t\right)\right)dt \ \left(t \mapsto \frac{\pi}{4}-t\right)$$

$$=\int_0^{\frac{\pi}{4}} \ln\left(1+\frac{1-\tan t}{1+\tan t}\right)dt = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan t}\right)dt = \int_0^{\frac{\pi}{4}} \{\ln 2-\ln(1+\tan t)\}dt$$

$$2I=\int_0^{\frac{\pi}{4}} \{\ln(1+\tan t)+(\ln 2-\ln(1+\tan t))\}dt = \int_0^{\frac{\pi}{4}} \ln 2 dt = \frac{\pi}{4}\ln 2$$

$$\therefore I=\int_0^1 \frac{\ln(x+1)}{x^2+1}dx = \frac{\pi}{8}\ln 2 \ ■$$

24. $I=\int_0^2 \frac{\ln(x+1)}{x^2-x+1}dx = \int_1^3 \frac{\ln t}{t^2-3t+3}dt \ (t=x+1,\ dt=dx)$

$$=\int_3^1 \frac{\ln(3u^{-1})}{9u^{-2}-9u^{-1}+3}\left(-\frac{3}{u^2}du\right) \ \left(u=\frac{3}{t},\ du=-\frac{3}{t^2}dt,\ dt=-\frac{3}{u^2}du\right)$$

$$=\int_1^3 \frac{\ln 3-\ln u}{u^2-3u+3}du = \int_1^3 \frac{\ln 3}{u^2-3u+3}du - \int_1^3 \frac{\ln u}{u^2-3u+3}du = \int_1^3 \frac{\ln 3}{u^2-3u+3}du - I$$

$$= \frac{\ln 3}{2}\int_1^3 \frac{1}{u^2-3u+3}du = \frac{\ln 3}{2}\int_1^3 \frac{1}{\left(u-\frac{3}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}du = \frac{\ln 3}{2}\left[\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x-3}{\sqrt{3}}\right)\right]_1^3$$

$$= \frac{\ln 3}{\sqrt{3}}\left(\frac{\pi}{3}+\frac{\pi}{6}\right) = \frac{\pi \ln 3}{2\sqrt{3}} \ \blacksquare$$

25. $I(\alpha) := \int_0^1 \frac{\ln(\alpha x^2+1)}{x+1}dx$, $\frac{d}{d\alpha}I(\alpha) = \int_0^1 \frac{\partial}{\partial \alpha}\frac{\ln(\alpha x^2+1)}{x+1}dx = \int_0^1 \frac{x^2}{(\alpha x^2+1)(x+1)}dx$

$$= \int_0^1\left(\frac{x-1}{(\alpha+1)(\alpha x^2+1)}+\frac{1}{(\alpha+1)(x+1)}\right)dx = \frac{1}{\alpha+1}\int_0^1\left(\frac{x}{\alpha x^2+1}-\frac{1}{\alpha x^2+1}+\frac{1}{x+1}\right)dx$$

$$= \frac{1}{\alpha+1}\left[\frac{1}{2\alpha}\ln\left|\alpha x^2+1\right|-\frac{1}{\sqrt{a}}\tan^{-1}(\sqrt{\alpha}\,x)+\ln|x+1|\right]_0^1$$

$$= \frac{1}{\alpha+1}\left(\frac{\ln(\alpha+1)}{2\alpha}-\frac{\tan^{-1}(\sqrt{\alpha}\,x)}{\sqrt{\alpha}}+\ln 2\right)$$

$$\therefore \int_0^1 \frac{\ln(x^2+1)}{x+1}dx = I(1) = \frac{1}{2}\int_0^1 \frac{\ln(\alpha+1)}{\alpha(\alpha+1)}d\alpha - \int_0^1 \frac{\tan^{-1}(\alpha)}{\sqrt{\alpha}(\alpha+1)}d\alpha + \int_0^1 \frac{\ln 2}{\alpha+1}d\alpha$$

$$= \frac{1}{2}\left(\int_0^1 \frac{\ln(\alpha+1)}{\alpha}d\alpha - \int_0^1 \frac{\ln(\alpha+1)}{\alpha+1}d\alpha\right) - 2\left[\frac{1}{2}\left(\tan^{-1}(\alpha)\right)^2\right]_0^1 + \ln 2\,[\ln(\alpha+1)]_0^1$$

$$= \frac{1}{2}\left(\int_0^1 \sum_{n=0}^{\infty}\frac{(-1)^n\alpha^n}{n+1}d\alpha - \left[\frac{1}{2}(\ln(\alpha+1))^2\right]_0^1\right) - \frac{\pi^2}{16}+(\ln 2)^2$$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\int_0^1 \alpha^n d\alpha - \frac{1}{2}(\ln 2)^2\right) - \frac{\pi^2}{16}+(\ln 2)^2$$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}\right) - \frac{\pi^2}{16}+\frac{3}{4}(\ln 2)^2 = \frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}-\sum_{n=1}^{\infty}\frac{1}{2n^2}\right) - \frac{\pi^2}{16}+\frac{3}{4}(\ln 2)^2$$

$$= \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}-\frac{\pi^2}{16}+\frac{3}{4}(\ln 2)^2 = \frac{1}{4}\zeta(2)-\frac{\pi^2}{16}+\frac{3}{4}(\ln 2)^2$$

$= \frac{3}{4}(\ln 2)^2 - \frac{\pi^2}{48}$ $\blacksquare$ ($\because$ 바젤 문제에 의해 $\zeta(2) = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$)

26. $I=\int_0^{\frac{\pi}{2}}\ln(\cos x)dx=\int_0^{\frac{\pi}{2}}\ln(\sin x)dx\ \ (x\mapsto\frac{\pi}{2}-x)$

$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\ln(\sin x\cos x)dx=\frac{1}{2}\int_0^{\frac{\pi}{2}}\ln\left(\frac{1}{2}\sin 2x\right)dx=\frac{1}{2}\int_0^{\frac{\pi}{2}}\ln(\sin 2x)dx-\frac{\pi}{4}\ln 2$

$=\frac{1}{4}\int_0^{\pi}\ln(\sin t)dt-\frac{\pi}{4}\ln 2\ \ (t=2x,\ dt=2dx)$

$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\ln(\sin t)dt-\frac{\pi}{4}\ln 2=\frac{1}{2}I-\frac{\pi}{4}\ln 2,\quad \frac{1}{2}I=-\frac{\pi}{4}\ln 2$

$\therefore I=\int_0^{\frac{\pi}{2}}\ln(\cos x)dx=-\frac{\pi}{2}\ln 2$ ■

27. sol 1) $I=\int_0^1\frac{\ln x}{x+1}dx=\int_0^1\frac{\ln x}{1-x}dx-\int_0^1\frac{2x\ln x}{1-x^2}dx$

$=\int_0^1\frac{\ln x}{1-x}dx-\frac{1}{2}\int_0^1\frac{\ln t}{1-t}dt\ \ (t=x^2,\ dt=2xdx)$

$=\frac{1}{2}\int_0^1\frac{\ln t}{1-t}dt$

$(1+x)^{-1}$의 이항급수를 구하고 양변을 적분하면

$(1+x)^{-1}=\sum_{n=0}^{\infty}\binom{-1}{n}\cdot x^n=\sum_{n=0}^{\infty}\prod_{i=0}^{n-1}(-1-i)\cdot\frac{x^n}{n!}=\sum_{n=0}^{\infty}(-1)^n\cdot x^n\quad(|x|<1)$

$\int(1+x)^{-1}dx=\ln|1+x|=\int\sum_{n=0}^{\infty}(-1)^n x^n dx+C=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1}+C$

$x=0$을 대입하면 $0=0+C$에서 $C=0$이다.

$\int_0^1\frac{\ln x}{x-1}dx=\int_{-1}^0\frac{\ln(t+1)}{t}dt=\int_{-1}^0\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}t^n dt=\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}t^{n+1}\right]_{-1}^0$

$=\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}0^{n+1}\right]-\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}(-1)^{n+1}\right]=\sum_{n=0}^{\infty}\frac{1}{(n+1)^2}$

$$= \zeta(2) = \frac{\pi^2}{6} \quad (\because \text{ 바젤 문제에 의해 } \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6})$$

$$\therefore I = \frac{1}{2}\int_0^1 \frac{\ln t}{1-t}dt = -\frac{\pi^2}{12}$$

sol 2) $\int_0^1 x^a dx = \frac{1}{a+1}$의 양변을 $a$로 미분하면

$$\frac{d}{da}\int_0^1 x^a dx = \int_0^1 \frac{\partial}{\partial a} x^a dx = \int_0^1 x^a \ln x dx = -\frac{1}{(a+1)^2}$$

한편 $|x| < 1$일 때 $\sum_{a=0}^{\infty} (-1)^a x^a = \frac{1}{1+x}$ 임을 이용하면

$$\sum_{a=0}^{\infty} (-1)^a \int_0^1 x^a \ln x dx = \int_0^1 \sum_{a=0}^{\infty} (-1)^a x^a \ln x dx = -\sum_{a=0}^{\infty} \frac{(-1)^a}{(a+1)^2}$$

$$\therefore \int_0^1 \frac{\ln x}{x+1} dx = -\sum_{a=0}^{\infty} \frac{(-1)^a}{(a+1)^2} = \sum_{a=1}^{\infty} \frac{(-1)^a}{a^2} = \sum_{a=1}^{\infty} \frac{1}{2a^2} - \sum_{a=1}^{\infty} \frac{1}{a^2}$$

$$= -\frac{1}{2}\sum_{a=1}^{\infty} \frac{1}{a^2} = -\frac{1}{2}\zeta(2) = -\frac{\pi^2}{12} \ \blacksquare$$

28. $n \in \mathbb{N} \cup \{0\}$, $a_n := \int_0^{\frac{\pi}{2}} \cos^n x \cos nx dx$

$$a_n = \int_0^{\frac{\pi}{2}} \cos^n x \{\cos(n+1)x \cos x + \sin(n+1)x \sin x\} dx$$

$$= \int_0^{\frac{\pi}{2}} \cos^{n+1} x \cos(n+1)x dx + \int_0^{\frac{\pi}{2}} \cos^n x \sin(n+1)x \sin x dx$$

$$= a_{n+1} + \left[\sin(n+1)x \cdot \frac{-1}{n+1} \cos^{n+1} x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \cos(n+1)x(-\cos^{n+1} x) dx$$

$$= 2a_{n+1}$$

$$a_0 = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2},\ \ a_1 = \int_0^{\frac{\pi}{2}} \cos^2 dx = \left[\frac{1}{2}x + \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

$$a_n = \int_0^{\frac{\pi}{2}} \cos^n x \cos nx dx = \pi \cdot \left(\frac{1}{2}\right)^{n+1}$$

$$\therefore a_{2022} = \int_0^{\frac{\pi}{2}} \cos^{2022} x \cos 2022x dx = \frac{\pi}{2^{2023}} \ \blacksquare$$

29. sol 1) $t = -\ln x,\ x = e^{-t},\ dx = -e^{-t}dt$

$$\int_0^1 (\ln x)^{2022} dx = \int_0^\infty t^{2022} e^{-t} dt = \Gamma(2023) = 2022! \ \blacksquare$$

sol 2) $J_n := \int_0^1 (-\ln x)^n dx = \left[x \cdot (-\ln x)^n\right]_0^1 - \int_0^1 x \cdot n(-\ln x)^{n-1} \cdot \frac{-1}{x} dx$

$$= 0 - \lim_{x\to 0}\{x \cdot (-\ln x)^n\} + n\int_0^1 (-\ln x)^{n-1} dx = nJ(n-1)$$

$(\because x = e^{-t},\ \lim_{x\to 0}\{x \cdot (-\ln x)^n\} = \lim_{t\to\infty} \frac{t^n}{e^t} = 0$ by l'Hospital's rule)

$$J_n = nJ_{n-1} = n(n-1)J_{n-2} = \ \cdots \ = n(n-1)\cdots 1 \cdot J_0 = n!$$

$$\therefore \int_0^1 (\ln x)^{2022} dx = J_{2022} = 2022! \ \blacksquare$$

30. $\int_1^e \frac{\ln x}{\sqrt{x}} dx = \left[\ln x \cdot 2\sqrt{x}\right]_1^e - \int_1^e \frac{2\sqrt{x}}{x} dx = 2\sqrt{e} - \left[4\sqrt{x}\right]_1^e = 4 - 2\sqrt{e} \ \blacksquare$

31. $I(\alpha) := \int_0^{\frac{\pi}{2}} \ln(a + \tan^2 x) dx,\ \ \frac{d}{d\alpha} I(\alpha) = \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial \alpha} \ln(a + \tan^2 x) dx = \int_0^{\frac{\pi}{2}} \frac{1}{a + \tan^2 x} dx$

$$= \int_0^\infty \frac{1}{(t^2+1)(a+t^2)} dt = \frac{1}{a-1}\int_0^\infty \left(\frac{1}{t^2+1} - \frac{1}{t^2+a}\right) dt \ \ (t = \tan x,\ dt = \sec^2 x dx)$$

$$= \frac{1}{a-1}\left[\tan^{-1} t - \frac{1}{\sqrt{a}} \tan^{-1}\left(\frac{t}{\sqrt{a}}\right)\right]_0^\infty = \frac{\pi}{2}\left(\frac{1}{a-1} - \frac{1}{\sqrt{a}(a-1)}\right)$$

$$I(\alpha) = \int \frac{\pi}{2}\left(\frac{1}{a-1} - \frac{1}{\sqrt{a}(a-1)}\right)da = \frac{\pi}{2}\int\left(\frac{1}{a-1} - \frac{1}{\sqrt{a}(a-1)}\right)dx$$

$$= \frac{\pi}{2}\left(\ln(a-1) + \ln\left(\frac{\sqrt{a}+1}{\sqrt{a}-1}\right)\right) = \pi\ln(\sqrt{a}+1)$$

$$(\because I(0) = 2\int_0^{\frac{\pi}{2}}\ln(\tan x)dx = 2\int_0^{\frac{\pi}{2}}\ln(\cot x)dx = -I(0) = 0)$$

$$\therefore \int_0^{\frac{\pi}{2}}\ln(2+\tan^2 x)dx = \pi\ln(\sqrt{2}+1) \blacksquare$$

32. $I(\alpha) := \int_0^\infty \frac{\alpha x - \sin(\alpha x)}{x^3(x^2+1)}dx$, $\frac{d^3}{d\alpha^3}I(\alpha) = \int_0^\infty \frac{\partial^3}{\partial\alpha^3}\frac{\alpha x - \sin(\alpha x)}{x^3(x^2+1)}dx$

$$= \int_0^\infty \frac{\partial^2}{\partial\alpha^2}\frac{1-\cos(\alpha x)}{x^2(x^2+1)}dx = \int_0^\infty \frac{\partial}{\partial\alpha}\frac{\sin(\alpha x)}{x(x^2+1)}dx = \int_0^\infty \frac{\cos(\alpha x)}{x^2+1}dx$$

$$J(\alpha) := \int_{-\infty}^\infty \frac{\cos\alpha x}{x^2+1}dx = \left[\frac{\sin(\alpha x)}{\alpha(x^2+1)}\right]_{-\infty}^\infty + \frac{2}{\alpha}\int_{-\infty}^\infty \frac{x\sin(\alpha x)}{(x^2+1)^2}dx$$

$$= \frac{2}{\alpha}\int_{-\infty}^\infty \frac{x\sin(\alpha x)}{(x^2+1)^2}dx, \quad \alpha J(\alpha) = 2\int_{-\infty}^\infty \frac{x\sin(\alpha x)}{(x^2+1)^2}dx$$

양변을 미분하면

$$\frac{d}{d\alpha}(\alpha J(\alpha)) = J(\alpha) + \alpha J'(\alpha) = 2\int_{-\infty}^\infty \frac{\partial}{\partial\alpha}\frac{x\sin(\alpha x)}{(x^2+1)^2}dx = 2\int_{-\infty}^\infty \frac{x^2\cos(\alpha x)}{(x^2+1)^2}dx$$

$$= 2\int_{-\infty}^\infty \frac{\cos(\alpha x)}{x^2+1}dx - 2\int_{-\infty}^\infty \frac{\cos(\alpha x)}{(x^2+1)^2}dx = 2J(\alpha) - 2\int_{-\infty}^\infty \frac{\cos(\alpha x)}{(x^2+1)^2}dx$$

$\alpha J'(\alpha) - J(\alpha) = -2\int_{-\infty}^\infty \frac{\cos(\alpha x)}{(x^2+1)^2}dx$, 다시 양변을 $\alpha$로 미분하면

$$\alpha J''(\alpha) = -2\int_{-\infty}^\infty \frac{\partial}{\partial\alpha}\frac{\cos(\alpha x)}{(x^2+1)^2}dx = 2\int_{-\infty}^\infty \frac{x\sin(\alpha x)}{(x^2+1)^2}dx = \alpha J(\alpha)$$

$J''(\alpha) = J(\alpha)$라는 미분방정식을 얻는다.

$J(\alpha)$의 이계도함수가 $J(\alpha)$와 동일해야 하므로 $J(\alpha)$는 지수함수 또는 삼각함수들의 합으로 구성되어 있어야 하고, 상수 $C_1$, $C_2$, $C_3$, $C_4$에 대하여
$J(\alpha)=C_1e^{f(\alpha)}+C_2e^{-f(\alpha)}$와 $J(\alpha)=C_3\sin g(\alpha)+C_4\cos g(\alpha)$의 형태를 모두 시험해보면 다음과 같은 해를 얻는다. (오일러공식 $e^{ix}=\cos x+i\sin x$를 적용하면 두 경우가 동일함을 알 수 있다.)

$$J(\alpha)=C_1e^{\alpha}+C_2e^{-\alpha}\quad(\alpha>0)$$

한편 $J(0)=C_1+C_2=\int_{-\infty}^{\infty}\frac{1}{x^2+1}dx=\left[\tan^{-1}x\right]_{-\infty}^{\infty}=\pi$,

$J(\alpha)=\frac{2}{\alpha}\int_{-\infty}^{\infty}\frac{x\sin(\alpha x)}{(x^2+1)^2}dx$에서 $\lim_{\alpha\to\infty}J(\alpha)=0$, $C_1=0$이고 $C_2=\pi$이다.

$\therefore J(\alpha)=\frac{\pi}{e^{\alpha}}$

$\frac{d^3}{d\alpha^3}I(\alpha)=\int_0^{\infty}\frac{\cos(\alpha x)}{x^2+1}dx=\frac{1}{2}J(\alpha)=\frac{\pi}{2e^{\alpha}}$이고

$I(0)=I'(0)=I''(0)=0$이므로

$I''(\alpha)=\frac{\pi}{2}(1-e^{-\alpha})$, $I'(\alpha)=\frac{\pi}{2}(\alpha+e^{-\alpha}-1)$, $I(\alpha)=\frac{\pi}{2}\left(\frac{1}{2}\alpha^2-e^{-\alpha}-\alpha+1\right)$이다.

한편 $x=2t$, $dx=2dt$의 치환을 하면

$$\int_0^{\infty}\frac{x-\sin x}{x^3(x^2+4)}dx=\int_0^{\infty}\frac{2t-\sin(2t)}{8t^3(4t^2+4)}\cdot 2dt=\frac{1}{16}\int_0^{\infty}\frac{2t-\sin(2t)}{t^3(t^2+1)}dt=\frac{1}{16}I(2)$$

$=\frac{\pi}{32}(1-e^{-2})$ ■

33. $u=\tan\frac{x}{2}$, $du=\frac{1}{2}\sec^2\frac{x}{2}dx$

$$\int_0^{\pi}\frac{1-\sin x}{1+\sin x}dx=\int_0^{\infty}\frac{2\left(1-\frac{2u}{u^2+1}\right)}{(u^2+1)\left(\frac{2u}{u^2+1}+1\right)}du=\int_0^{\infty}\frac{2(u-1)^2}{(u^2+1)(u+1)^2}du$$

$$= 2\int_0^{\infty}\left(\frac{2}{(u+1)^2}-\frac{1}{u^2+1}\right)du = 2\left[-\frac{2}{u+1}-\tan^{-1}u\right]_0^{\infty} = 4-\pi \ \blacksquare$$

34. $\int_0^{\frac{3}{5}}\frac{e^x(2-x^2)}{(1-x)\sqrt{1-x^2}}dx = \int_0^{\frac{3}{5}}\left(\frac{e^x}{(1-x)\sqrt{1-x^2}}+e^x\sqrt{\frac{1+x}{1-x}}\right)dx$

$$= \int_0^{\frac{3}{5}}\left(e^x\left(\sqrt{\frac{1+x}{1-x}}\right)'+(e^x)'\sqrt{\frac{1+x}{1-x}}\right)dx = \left[e^x\sqrt{\frac{1+x}{1-x}}\right]_0^{\frac{3}{5}} = 2e^{\frac{3}{5}}-1 \ \blacksquare$$

35. $\int_0^{\frac{\pi}{2}}\sin 2022x \cdot \sin^{2020}x dx = \int_0^{\frac{\pi}{2}}\sin(2021x+x)\cdot\sin^{2020}x dx$

$$= \int_0^{\frac{\pi}{2}}\sin 2021x\cdot\cos x\cdot\sin^{2020}x dx + \int_0^{\frac{\pi}{2}}\cos 2021x\cdot\sin^{2021}x dx$$

$$= \sin 2021x\cdot\frac{1}{2021}\sin^{2021}x - \frac{2021}{2021}\int_0^{\frac{\pi}{2}}\cos 2021x\cdot\sin^{2021}x dx + \int_0^{\frac{\pi}{2}}\cos 2021x\cdot\sin^{2021}x dx$$

$$= \left[\frac{1}{2021}\sin 2021x\cdot\sin^{2021}x dx\right]_0^{\frac{\pi}{2}} = \frac{1}{2021} \ \blacksquare$$

36. $t = x^2+\frac{1}{x^2}-1,\ dt = 2\left(x-\frac{1}{x^3}\right)dx$

$$\int_1^{\sqrt{2}}\frac{x^4-1}{x^2\sqrt{x^4-x^2+1}}dx = \int_1^{\sqrt{2}}\frac{x^4-1}{x^3\sqrt{x^2+\frac{1}{x^2}-1}}dx = \int_0^{\sqrt{2}}\frac{x-1/x^3}{\sqrt{x^2+\frac{1}{x^2}-1}}dx$$

$$= \frac{1}{2}\int_1^{\frac{3}{2}}\frac{1}{\sqrt{t}}dt = [\sqrt{t}]_1^{\frac{3}{2}} = \frac{\sqrt{6}}{2}-1 \ \blacksquare$$

37. $u=\cos x,\ du = -\sin x dx$

$$\int_0^{\frac{\pi}{3}}\frac{\sin 2x}{2+\cos x}dx = (-2)\int_0^{\frac{\pi}{3}}\frac{\cos x(-\sin x)}{2+\cos x}dx = (-2)\int_1^{\frac{1}{2}}\frac{u}{2+u}du = 2\int_{\frac{1}{2}}^1\left(1-\frac{2}{u+2}\right)du$$

$$= 2[u-2\ln|u+2|]_{\frac{1}{2}}^1 = 2(1-\ln 9)-2\left(\frac{1}{2}-2\ln\frac{5}{2}\right) = 1-\ln\left(\frac{1296}{625}\right) \ \blacksquare$$

38. (19)의 reduction formula를 이용하면

$$\int \tan^n x dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x dx \text{ 에서}$$

$$\int_0^{\frac{\pi}{4}} \tan^9 x dx = \left[\frac{\tan^8 x}{8}\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan^7 x dx = \frac{1}{8} - \left[\frac{\tan^6 x}{6}\right]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \tan^5 x dx$$

$$= \frac{1}{8} - \frac{1}{6} + \left[\frac{\tan^4 x}{4}\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan^3 x dx = \frac{1}{8} - \frac{1}{6} + \frac{1}{4} - \left[\frac{\tan^2 x}{2}\right]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \tan x dx$$

$$= \frac{1}{8} - \frac{1}{6} + \frac{1}{4} - \frac{1}{2} + [-\ln|\cos x|]_0^{\frac{\pi}{4}} = \frac{1}{8} - \frac{1}{6} + \frac{1}{4} - \frac{1}{2} + \ln\sqrt{2} = \frac{1}{2}\ln 2 - \frac{7}{24} \blacksquare$$

39. $I = \int_0^1 \frac{x-1}{(1+x^3)\ln x} dx = \frac{1}{2}\int_0^\infty \frac{x-1}{(1+x^3)\ln x} dx \ \left(x \mapsto \frac{1}{x}\right)$

$$J(a) := \int_0^\infty \frac{x^a - 1}{(1+x^3)\ln x} dx, \quad J'(a) = \frac{\partial}{\partial a} J(a) = \int_0^\infty \frac{x^a}{1+x^3} dx = \frac{\pi}{3}\csc\left(\frac{\pi(a+1)}{3}\right)$$

$$\therefore I = \frac{1}{2} J(1) = \frac{1}{2}\int_0^1 J'(a) da = \frac{\pi}{6}\int_0^1 \csc\left(\frac{\pi(a+1)}{3}\right) da = \frac{\ln 3}{2} \blacksquare$$

40. sol 1) $u = \frac{\sin t}{\sqrt{\cos t}}$, $du = \frac{1+\cos^2 t}{2\cos^{3/2} t} dt$

$$\int_0^{\frac{\pi}{2}} \frac{(1+\sec^2 t)\sqrt{\sec t}}{(1+\sec t)^2 - 2} dt = \int_0^{\frac{\pi}{2}} \frac{1+\cos^2 t}{\sqrt{\cos t}\{(\cos t + 1)^2 - 2\cos^2 t\}} dt$$

$$= \int_0^{\frac{\pi}{2}} \frac{1+\cos^2 t}{\sqrt{\cos t}(1+2\cos t - \cos^2 t)} dt = 2\int_0^{\frac{\pi}{2}} \frac{\cos t}{1+2\cos t - \cos^2 t} \cdot \frac{1+\cos^2 t}{2\cos^{3/2} t} dt$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{1}{\frac{1}{\cos t} + 2 - \cos t} \cdot \frac{1+\cos^2 t}{2\cos^{3/2} t} dt = 2\int_0^{\frac{\pi}{2}} \frac{1}{\frac{1-\cos^2 t}{\cos t} + 2} \cdot \frac{1+\cos^2 t}{2\cos^{3/2} t} dt$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{1}{\frac{\sin^2 t}{\cos t} + 2} \cdot \frac{1+\cos^2 t}{2\cos^{3/2} t} dt = 2\int_0^\infty \frac{1}{u^2+2} du = 2\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_0^\infty = \frac{\pi}{\sqrt{2}} \blacksquare$$

sol 2) $\sec t = x^2$, $\sec t \tan t dt = 2x dx$, $u = \sqrt{x^2 - \frac{1}{x^2}}$, $du = \frac{x^4+1}{x^2\sqrt{x^4-1}}$

$$I = \int_0^{\frac{\pi}{2}} \frac{(1+\sec^2 t)\sqrt{\sec t}}{(1+\sec t)^2 - 2} dt = 2\int_1^{\infty} \frac{x^4+1}{\sqrt{x^4-1}(x^4+2x^2-1)} dx = 2\int_0^{\infty} \frac{1}{u^2+2} du$$

$$= 2\left[\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_0^{\infty} = \frac{\pi}{\sqrt{2}} \ \blacksquare$$

41. $$\int_0^{\frac{\pi}{2}} \frac{\cos x}{2-\sin 2x} dx = \int_0^{\frac{\pi}{2}} \frac{\cos x}{2 - 2\sin x \cos x} dx = \int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\cos^2 x - 2\sin x \cos x + \sin^2 x} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{1+(\cos x - \sin x)^2} dx = \frac{1}{2}\left[\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+(\cos x - \sin x)^2} dx + \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+(\cos x - \sin x)^2} dx\right]$$

$$\left(\because \ x \mapsto \frac{\pi}{2} - x \ : \ \int_0^{\frac{\pi}{2}} \frac{\cos x}{1+(\cos x - \sin x)^2} dx = \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+(\cos x - \sin x)^2} dx\right)$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\cos x + \sin x}{1+(\cos x - \sin x)^2} dx = -\frac{1}{2}\left[\tan^{-1}(\cos x - \sin x)\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4} \ \blacksquare$$

42. $$\int_0^{\infty} (\ln(e^x+1) - x) dx = \int_0^{\infty} \ln\left(\frac{e^x+1}{e^x}\right) dx = \int_0^{\infty} \ln(1+e^{-x}) dx$$

$$= \int_0^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} e^{-nx}}{n} dx = \sum_{n=1}^{\infty} \int_0^{\infty} \frac{(-1)^{n-1} e^{-nx}}{n} dx = \sum_{n=1}^{\infty} \left[\frac{(-1)^n e^{-nx}}{n^2}\right]_0^{\infty}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2} - 2\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\zeta(2)}{2} = \frac{\pi^2}{12} \ \blacksquare$$

43. $$I = \int_0^{\infty} \frac{x}{e^x+1} dx = \int_1^{\infty} \frac{\ln t}{t(1+t)} dt \ \ \left(x = \ln t, \ dx = \frac{1}{t} dt\right)$$

$$= -\int_0^1 \frac{\ln u}{1+u} du \ \ \left(u = \frac{1}{t}, \ du = -\frac{1}{u^2} du\right)$$

$$= \int_0^1 \frac{\ln u}{1-u} du - \int_0^1 \frac{2u \ln u}{1-u^2} du = \int_0^1 \frac{\ln u}{1-u} du - \frac{1}{2}\int_0^1 \frac{\ln y}{1-y} dy \ \ (y = u^2, \ dy = 2u du)$$

$$=\frac{1}{2}\int_0^1 \frac{\ln u}{1-u}du$$

$(1+x)^{-1}$의 이항급수를 구하고 양변을 적분하면

$$(1+x)^{-1}=\sum_{n=0}^{\infty}\binom{-1}{n}\cdot x^n=\sum_{n=0}^{\infty}\prod_{i=0}^{n-1}(-1-i)\cdot\frac{x^n}{n!}=\sum_{n=0}^{\infty}(-1)^n\cdot x^n \quad (|x|<1)$$

$$\int(1+x)^{-1}dx=\ln|1+x|=\int\sum_{n=0}^{\infty}(-1)^n x^n dx+C=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1}+C$$

$x=0$을 대입하면 $0=0+C$에서 $C=0$이다.

$$\int_0^1\frac{\ln x}{x-1}dx=\int_{-1}^0\frac{\ln(t+1)}{t}dt=\int_{-1}^0\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}t^n dt=\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}t^{n+1}\right]_{-1}^0$$

$$=\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}0^{n+1}\right]-\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}(-1)^{n+1}\right]=\sum_{n=0}^{\infty}\frac{1}{(n+1)^2}$$

$$=\zeta(2)=\frac{\pi^2}{6}\ (\because \text{ 바젤 문제에 의해 } \zeta(2)=\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6})$$

$$I=\int_0^{\infty}\frac{x}{e^x+1}dx=\int_1^{\infty}\frac{\ln t}{t(1+t)}dt=\frac{1}{2}\int_0^1\frac{\ln u}{1-u}du=\frac{\pi^2}{12}\ \blacksquare$$

44. $$\int_0^{\frac{\pi}{2}}\frac{\ln(\tan x)}{1-\tan x+\tan^2 x}dx=\int_0^{\infty}\frac{\ln t}{(1-t+t^2)(1+t^2)}dt\ (t=\tan x,\ dt=\sec^2 x dx)$$

$$=\int_0^1\frac{\ln t}{(1-t+t^2)(1+t^2)}dt+\int_1^{\infty}\frac{\ln t}{(1-t+t^2)(1+t^2)}dt$$

$$=\int_0^1\frac{\ln t}{(1-t+t^2)(1+t^2)}dt-\int_0^1\frac{u^2\ln u}{(1-u+u^2)(1+u^2)}du\ \left(u=\frac{1}{t},\ du=-\frac{1}{t^2}dt\right)$$

$$=\int_0^1\frac{(1-u^2)\ln u}{(1-u+u^2)(1+u^2)}du=\int_0^1\frac{2u\ln u}{1+u^2}du-\int_0^1\frac{(2u-1)\ln u}{1-u+u^2}du$$

$$=\left[\ln u\ln(u^2+1)\right]_0^1-\int_0^1\frac{1}{u}\ln(u^2+1)du-\left[\ln u\ln(1-u+u^2)\right]_0^1+\int_0^1\frac{1}{u}\ln(1-u+u^2)du$$

$$= -\int_0^1 \frac{\ln(1+u^2)}{u}du + \int_0^1 \frac{\ln(1-u+u^2)}{u}du$$

$$= -\int_0^1 \frac{\ln(1+u^2)}{u^2}udu + \int_0^1 \frac{\ln(1+u^3)}{u}du - \int_0^1 \frac{\ln(1+u)}{u}du$$

$$= -\frac{1}{2}\int_0^1 \frac{\ln(1+y)}{y}dy + \int_0^1 \frac{\ln(1+u^3)}{u^3}u^2du - \int_0^1 \frac{\ln(1+u)}{u}du \ \ (y=u^2,\ dy=2udu)$$

$$= -\frac{3}{2}\int_0^1 \frac{\ln(1+u)}{u}du + \frac{1}{3}\int_0^1 \frac{\ln(1+z)}{z}dz \ \ (z=u^3,\ dz=3u^2du)$$

$$= -\frac{7}{6}\int_0^1 \frac{\ln(1+u)}{u}du = -\frac{7}{6}\int_0^1 \sum_{n=0}^{\infty}\frac{(-1)^n u^n}{n+1}du = -\frac{7}{6}\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\int_0^1 u^n du$$

$$= -\frac{7}{6}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2} = -\frac{7}{6}\left(\sum_{n=1}^{\infty}\frac{1}{n^2} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}\right) = -\frac{7}{12}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$= -\frac{7}{12}\zeta(2) = -\frac{7\pi^2}{72}$$ ■ ($\because$ 바젤 문제에 의해 $\zeta(2)=\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$)

45. $t = x - \frac{1}{x}$, $dt = \left(1+\frac{1}{x^2}\right)dx$, $\frac{dx}{1+x^2} = \frac{x^2}{(1+x^2)^2}dt = \frac{dt}{(x+1/x)^2} = \frac{dt}{t^2+4}$

$$I = \int_{-\infty}^{\infty}\frac{\cos\left(x-\frac{1}{x}\right)}{1+x^2}dx = 2\int_0^{\infty}\frac{\cos\left(x-\frac{1}{x}\right)}{1+x^2}dx = 2\int_{-\infty}^{\infty}\frac{\cos t}{t^2+4}dt$$

$$J(\alpha) := \int_{-\infty}^{\infty}\frac{\cos\alpha x}{x^2+1}dx = \left[\frac{\sin(\alpha x)}{\alpha(x^2+1)}\right]_{-\infty}^{\infty} + \frac{2}{\alpha}\int_{-\infty}^{\infty}\frac{x\sin(\alpha x)}{(x^2+1)^2}dx$$

$$= \frac{2}{\alpha}\int_{-\infty}^{\infty}\frac{x\sin(\alpha x)}{(x^2+1)^2}dx, \quad \alpha J(\alpha) = 2\int_{-\infty}^{\infty}\frac{x\sin(\alpha x)}{(x^2+1)^2}dx$$

양변을 미분하면

$$\frac{d}{d\alpha}(\alpha J(\alpha)) = J(\alpha) + \alpha J'(\alpha) = 2\int_{-\infty}^{\infty}\frac{\partial}{\partial\alpha}\frac{x\sin(\alpha x)}{(x^2+1)^2}dx = 2\int_{-\infty}^{\infty}\frac{x^2\cos(\alpha x)}{(x^2+1)^2}dx$$

$$= 2\int_{-\infty}^{\infty}\frac{\cos(\alpha x)}{x^2+1}dx - 2\int_{-\infty}^{\infty}\frac{\cos(\alpha x)}{(x^2+1)^2}dx = 2J(\alpha) - 2\int_{-\infty}^{\infty}\frac{\cos(\alpha x)}{(x^2+1)^2}dx$$

$\alpha J'(\alpha) - J(\alpha) = -2\int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{(x^2+1)^2} dx$, 다시 양변을 $\alpha$로 미분하면

$$\alpha J''(\alpha) = -2\int_{-\infty}^{\infty} \frac{\partial}{\partial \alpha} \frac{\cos(\alpha x)}{(x^2+1)^2} dx = 2\int_{-\infty}^{\infty} \frac{x\sin(\alpha x)}{(x^2+1)^2} dx = \alpha J(\alpha)$$

$J''(\alpha) = J(\alpha)$라는 미분방정식을 얻는다.

$J(\alpha)$의 이계도함수가 $J(\alpha)$와 동일해야 하므로 $J(\alpha)$는 지수함수 또는 삼각함수들의 합으로 구성되어 있어야 하고, 상수 $C_1$, $C_2$, $C_3$, $C_4$에 대하여 $J(\alpha) = C_1 e^{f(\alpha)} + C_2 e^{-f(\alpha)}$와 $J(\alpha) = C_3 \sin g(\alpha) + C_4 \cos g(\alpha)$의 형태를 모두 시험해보면 다음과 같은 해를 얻는다. (오일러공식 $e^{ix} = \cos x + i\sin x$를 적용하면 두 경우가 동일함을 알 수 있다.)

$$J(\alpha) = C_1 e^{\alpha} + C_2 e^{-\alpha} \quad (\alpha \geq 0)$$

한편 $J(0) = C_1 + C_2 = \int_{-\infty}^{\infty} \frac{1}{x^2+1} dx = [\tan^{-1} x]_{-\infty}^{\infty} = \pi$,

$J(\alpha) = \frac{2}{\alpha}\int_{-\infty}^{\infty} \frac{x\sin(\alpha x)}{(x^2+1)^2} dx$에서 $\lim_{\alpha\to\infty} J(\alpha) = 0$, $C_1 = 0$이고 $C_2 = \pi$이다.

$$J(\alpha) = \int_{-\infty}^{\infty} \frac{\cos\alpha x}{x^2+1} dx = \frac{\pi}{e^{\alpha}}$$

$$\therefore I = 2\int_{-\infty}^{\infty} \frac{\cos t}{t^2+4} dt = 4\int_{-\infty}^{\infty} \frac{\cos 2u}{4u^2+4} du \;\; (t = 2u,\; dt = 2du)$$

$$= J(2) = \frac{\pi}{e^2}$$ ■

46. $I(\alpha) := \int_{-\infty}^{\infty} \frac{\cos\alpha x}{x^2+1} dx = \left[\frac{\sin(\alpha x)}{\alpha(x^2+1)}\right]_{-\infty}^{\infty} + \frac{2}{\alpha}\int_{-\infty}^{\infty} \frac{x\sin(\alpha x)}{(x^2+1)^2} dx$

$= \frac{2}{\alpha}\int_{-\infty}^{\infty} \frac{x\sin(\alpha x)}{(x^2+1)^2} dx$, $\quad \alpha I(\alpha) = 2\int_{-\infty}^{\infty} \frac{x\sin(\alpha x)}{(x^2+1)^2} dx$

양변을 미분하면

$$\frac{d}{d\alpha}(\alpha I(\alpha)) = I(\alpha) + \alpha I'(\alpha) = 2\int_{-\infty}^{\infty} \frac{\partial}{\partial \alpha} \frac{x\sin(\alpha x)}{(x^2+1)^2} dx = 2\int_{-\infty}^{\infty} \frac{x^2\cos(\alpha x)}{(x^2+1)^2} dx$$

$$= 2\int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{x^2+1} dx - 2\int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{(x^2+1)^2} dx = 2I(\alpha) - 2\int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{(x^2+1)^2} dx$$

$\alpha I'(\alpha) - I(\alpha) = -2\int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{(x^2+1)^2} dx$, 다시 양변을 $\alpha$로 미분하면

$$\alpha I''(\alpha) = -2\int_{-\infty}^{\infty} \frac{\partial}{\partial \alpha} \frac{\cos(\alpha x)}{(x^2+1)^2} dx = 2\int_{-\infty}^{\infty} \frac{x\sin(\alpha x)}{(x^2+1)^2} dx = \alpha I(\alpha)$$

$I''(\alpha) = I(\alpha)$라는 미분방정식을 얻는다.

$I(\alpha)$의 이계도함수가 $I(\alpha)$와 동일해야 하므로 $I(\alpha)$는 지수함수 또는 삼각함수들의 합으로 구성되어 있어야 하고, 상수 $C_1$, $C_2$, $C_3$, $C_4$에 대하여
$I(\alpha) = C_1 e^{f(\alpha)} + C_2 e^{-f(\alpha)}$와 $I(\alpha) = C_3 \sin g(\alpha) + C_4 \cos g(\alpha)$의 형태를 모두 시험해보면 다음과 같은 해를 얻는다. (오일러공식 $e^{ix} = \cos x + i\sin x$를 적용하면 두 경우가 동일함을 알 수 있다.)

$$I(\alpha) = C_1 e^{\alpha} + C_2 e^{-\alpha} \quad (\alpha \ge 0)$$

한편 $I(0) = C_1 + C_2 = \int_{-\infty}^{\infty} \frac{1}{x^2+1} dx = [\tan^{-1} x]_{-\infty}^{\infty} = \pi$,

$I(\alpha) = \frac{2}{\alpha}\int_{-\infty}^{\infty} \frac{x\sin(\alpha x)}{(x^2+1)^2} dx$에서 $\lim_{\alpha\to\infty} I(\alpha) = 0$, $C_1 = 0$이고 $C_2 = \pi$이다.

$I(\alpha) = \frac{\pi}{e^{\alpha}}$, $\therefore \int_0^{\infty} \frac{\cos x}{x^2+1} dx = \frac{1}{2} I(1) = \frac{\pi}{2e}$ ■

47. $I(\alpha) := \int_0^{\infty} \frac{x\sin(\alpha x)}{x^2+1} dx = \int_0^{\infty} \frac{x^2 \sin(\alpha x)}{x(x^2+1)} dx \ (\alpha > 0)$

$$= \int_0^{\infty} \frac{(1+x^2-1)\sin(\alpha x)}{x(x^2+1)} dx = \int_0^{\infty} \frac{\sin(\alpha x)}{x} dx - \int_0^{\infty} \frac{\sin(\alpha x)}{x(1+x^2)} dx$$

$$= \int_0^{\infty} \frac{\sin t}{t} dt - \int_0^{\infty} \frac{\sin(\alpha x)}{x(1+x^2)} dx \ (t = \alpha x,\ dt = \alpha dx)$$

$$= \frac{\pi}{2} - \int_0^\infty \frac{\sin(\alpha x)}{x(1+x^2)} dx \quad (※)$$

$$\frac{d}{d\alpha} I(\alpha) = -\int_0^\infty \frac{\partial}{\partial\alpha} \frac{\sin(\alpha x)}{x(1+x^2)} dx = -\int_0^\infty \frac{\cos(\alpha x)}{1+x^2} dx$$

$$\frac{d^2}{d\alpha^2} I(\alpha) = -\int_0^\infty \frac{\partial}{\partial\alpha} \frac{\cos(\alpha x)}{1+x^2} dx = \int_0^\infty \frac{x\sin(\alpha x)}{1+x^2} dx = I(\alpha)$$

$I''(\alpha) = I(\alpha)$라는 미분방정식을 얻는다.

$I(\alpha)$의 이계도함수가 $I(\alpha)$와 동일해야 하므로 $I(\alpha)$는 지수함수 또는 삼각함수들의 합으로 구성되어 있어야 하고, 상수 $C_1$, $C_2$, $C_3$, $C_4$에 대하여
$I(\alpha) = C_1 e^{f(\alpha)} + C_2 e^{-f(\alpha)}$와 $I(\alpha) = C_3 \sin g(\alpha) + C_4 \cos g(\alpha)$의 형태를 모두 시험해보면 다음과 같은 해를 얻는다. (오일러공식 $e^{ix} = \cos x + i\sin x$를 적용하면 두 경우가 동일함을 알 수 있다.)

$$I(\alpha) = C_1 e^{\alpha} + C_2 e^{-\alpha} \quad (\alpha > 0)$$

한편

$$\lim_{\alpha\to 0+} I(\alpha) = \lim_{\alpha\to 0+}\left(\frac{\pi}{2} - \int_0^\infty \frac{\sin(\alpha x)}{x(1+x^2)} dx\right) = \frac{\pi}{2}$$에서 $C_1 + C_2 = \frac{\pi}{2}$,

$$\lim_{\alpha\to 0+} I'(\alpha) = \lim_{\alpha\to 0+}\left(-\int_0^\infty \frac{\cos(\alpha x)}{1+x^2} dx\right) = -\frac{\pi}{2}$$에서 $C_1 - C_2 = -\frac{\pi}{2}$를 얻는다.

따라서 $C_1 = 0$, $C_2 = \frac{\pi}{2}$이고

$$I(\alpha) = \int_0^\infty \frac{x\sin(\alpha x)}{x^2+1} dx = \frac{\pi}{2e^{\alpha}},$$

$$\therefore \int_0^\infty \frac{x\sin x}{x^2+1} dx = I(1) = \frac{\pi}{2e} \quad ■$$

※ $J(\alpha) := \int_0^\infty \frac{\sin x}{x} e^{-\alpha x} dx$, $\frac{d}{d\alpha} J(\alpha) = \int_0^\infty \frac{\partial}{\partial\alpha} \frac{\sin x}{x} e^{-\alpha x} dx = -\int_0^\infty e^{-\alpha x} \sin x\, dx$

$$= -\left[-e^{-\alpha x}\cos x\right]_0^\infty + \int_0^\infty \alpha e^{-\alpha x} \cos x\, dx = -1 + \left[\alpha e^{-\alpha x} \sin x\right]_0^\infty + \int_0^\infty \alpha^2 e^{-\alpha x} \sin x\, dx$$

$$= -1-\alpha^2 J'(\alpha),\quad J'(\alpha) = -\frac{1}{1+\alpha^2}$$

$J(\alpha) = \int J'(\alpha)d\alpha = C - \tan^{-1}\alpha\ \ (C \in \mathbb{R}),\quad 0 = \lim_{\alpha\to\infty} J(\alpha) = C - \frac{\pi}{2}$ 에서 $C = \frac{\pi}{2}$ 이다.

$$\therefore \int_0^\infty \frac{\sin x}{x}dx = J(0) = \frac{\pi}{2}\ ■$$

48. $\int_0^\infty \frac{\ln(1+x)}{x(x^2+1)}dx = \int_0^1 \frac{\ln(1+x)}{x(x^2+1)}dx + \int_1^\infty \frac{\ln(1+x)}{x(x^2+1)}dx$

$$= \int_0^1 \frac{\ln(1+x)}{x(x^2+1)}dx + \int_1^0 \frac{t\ln(1+t^{-1})}{t^{-2}+1}\left(-\frac{1}{t^2}dt\right)\ \left(t = \frac{1}{x},\ dt = -\frac{1}{x^2}dx\right)$$

$$= \int_0^1 \frac{\ln(1+x)}{x(x^2+1)}dx + \int_0^1 \frac{t\ln(1+t^{-1})}{1+t^2}dt$$

$$= \int_0^1 \frac{\ln(1+x)}{x(x^2+1)}dx + \int_0^1 \frac{t\ln(1+t)}{1+t^2}dt - \int_0^1 \frac{t\ln t}{1+t^2}dt$$

$$= \int_0^1 \left(t + \frac{1}{t}\right)\frac{\ln(1+t)}{1+t^2}dt - \int_0^1 \frac{t\ln t}{1+t^2}dt$$

$$= \int_0^1 \frac{\ln(1+t)}{t}dt - \frac{1}{4}\int_0^1 \frac{\ln u}{1+u}du\ \ (u = t^2,\ du = 2tdt)$$

$$= \int_0^1 \frac{\ln(1+t)}{t}dt - \frac{1}{4}\left([\ln t\ln(1+t)]_0^1 - \int_0^1 \frac{\ln(1+t)}{t}dt\right) = \frac{5}{4}\int_0^1 \frac{\ln(1+t)}{t}dt$$

$$= \frac{5}{4}\int_0^1 \sum_{n=0}^\infty \frac{(-1)^n u^n}{n+1}du = \frac{5}{4}\sum_{n=0}^\infty \frac{(-1)^n}{n+1}\int_0^1 u^n du = \frac{5}{4}\sum_{n=0}^\infty \frac{(-1)^n}{(n+1)^2}$$

$$= \frac{5}{4}\left(\sum_{n=1}^\infty \frac{1}{n^2} - \frac{1}{2}\sum_{n=1}^\infty \frac{1}{n^2}\right) = \frac{5}{8}\sum_{n=1}^\infty \frac{1}{n^2} = \frac{5}{8}\zeta(2) = \frac{5\pi^2}{48}\ ■$$

($\because$ 바젤 문제에 의해 $\zeta(2) = \sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$)

※ $x=\tan t$의 치환을 한 후 $I(\alpha):=\int_0^{\frac{\pi}{2}}\frac{\ln(1+\alpha\tan t)}{\tan t}dt$로 정의하면 파인만 테크닉을 적용하여

$$I(\alpha)=\frac{\pi+2\alpha\ln\alpha}{2(1+\alpha^2)},\quad I(1)=\int_0^\infty\frac{\ln(1+x)}{x(1+x^2)}dx=\frac{5\pi^2}{48}$$

을 얻을 수 있다.

49. $v=\tan u,\ dv=\sec^2 u du$

$$I(\alpha):=\int_0^{\frac{\pi}{2}}\frac{\sin(u+\alpha\tan u)}{\sin u}du=\int_0^{\frac{\pi}{2}}\left(\cos(\alpha\tan u)+\frac{\sin(\alpha\tan u)}{\tan u}\right)du$$

$$=\int_0^\infty\frac{\cos(\alpha v)-\frac{\sin(\alpha v)}{v}}{1+v^2}dv,$$

$$\frac{d}{d\alpha}I(\alpha)=\int_0^\infty\frac{\partial}{\partial\alpha}\frac{\cos(\alpha v)-\frac{\sin(\alpha v)}{v}}{1+v^2}dv=\int_0^\infty\frac{\cos(\alpha v)-v\sin(\alpha v)}{1+v^2}dv$$

$$=\int_0^\infty\frac{\cos(\alpha v)}{1+v^2}dv-\int_0^\infty\frac{v\sin(\alpha v)}{1+v^2}dv=\frac{\pi}{2e}-\frac{\pi}{2e}=0\ (\because\ \#46,\ \#47)$$

$$\therefore I(\alpha)=C=I(0)=\frac{\pi}{2}$$

$$\therefore\int_0^{\frac{\pi}{2}}\frac{\sin(u+\sqrt{\pi}\tan u)}{\sin u}du=I(\sqrt{\pi})=\frac{\pi}{2}\ \blacksquare$$

50. $I(\alpha):=\int_0^\alpha\sin^2\left(x-\sqrt{\alpha^2-x^2}\right)dx=\int_{\frac{\pi}{2}}^0\sin^2(\alpha\cos t-\alpha\sin t)\cdot(-\alpha\sin t dt)$

$$=\int_0^{\frac{\pi}{2}}\sin^2(\alpha\cos t-\alpha\sin t)\alpha\sin t dt\ \ (x=\alpha\cos t,\ dx=-\alpha\sin t dt)$$

$$I(\alpha):=\int_0^\alpha\sin^2\left(x-\sqrt{\alpha^2-x^2}\right)dx=\int_0^{\frac{\pi}{2}}\sin^2(\alpha\sin t-\alpha\cos t)\alpha\cos t dt$$

$$= \int_0^{\frac{\pi}{2}} \sin^2(\alpha \cos t - \alpha \sin t)\alpha \cos t dt \ \ (x = \alpha \sin t,\ dx = \alpha \cos t dt)$$

$$I(\alpha) = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^2(\alpha(\sin t - \cos t))\alpha(\sin t + \cos t)dt$$

$$= \frac{1}{2}\int_{-\alpha}^{\alpha} \sin^2 u du \ \ (u = \alpha(\sin t - \cos t),\ du = \alpha(\cos t + \sin t)dt)$$

$$= \frac{1}{4}\int_{-\alpha}^{\alpha} (1 - \cos 2u)du = \frac{1}{4}\left[u - \frac{1}{2}\sin 2u\right]_{-\alpha}^{\alpha} = \frac{\alpha}{2} - \frac{1}{4}\sin 2\alpha$$

$$\therefore \int_0^{\pi} \sin^2\left(x - \sqrt{\pi^2 - x^2}\right)dx = I(\pi) = \frac{\pi}{2} \ \blacksquare$$

51. $I(\alpha) := \int_0^{\infty} e^{-x^2}\cos(\alpha x)dx$

$$\frac{d}{d\alpha}I(\alpha) = \int_0^{\infty} \frac{\partial}{\partial \alpha}e^{-x^2}\cos(\alpha x)dx = -\int_0^{\infty} xe^{-x^2}\sin(\alpha x)dx$$

$$= -\left(\left[-\frac{1}{2}e^{-x^2}\sin(\alpha x)\right]_0^{\infty} + \frac{\alpha}{2}\int_0^{\infty} e^{-x^2}\cos(\alpha x)dx\right) = -\frac{\alpha}{2}I(\alpha)$$

$I'(\alpha) = -\frac{\alpha}{2}I(\alpha)$의 미분방정식을 얻는다. 이를 풀기 위해 변수를 분리하고 양변을 적분하면

$$\frac{1}{I(\alpha)}dI(\alpha) = -\frac{\alpha}{2}d\alpha, \quad \int \frac{1}{I(\alpha)}dI(\alpha) = -\int \frac{\alpha}{2}d\alpha, \quad \ln|I(\alpha)| + C = -\frac{1}{4}\alpha^2$$

$I(0) = \int_0^{\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$ ($\because$ 가우스 적분) 이므로 $C = -\ln\left(\frac{\sqrt{\pi}}{2}\right)$

$$I(\alpha) = \int_0^{\infty} e^{-x^2}\cos(\alpha x)dx = \frac{\sqrt{\pi}}{2}e^{-\frac{\alpha^2}{4}}$$

$$\therefore \int_0^{\infty} e^{-x^2}\cos 2x dx = I(2) = \frac{\sqrt{\pi}}{2e} \ \blacksquare$$

52. $I(\alpha):=\int_0^{\frac{\pi}{2}}\cos(\alpha\tan x)dx$, $t=\tan x$, $dt=\sec^2 x dx$, $dx=\dfrac{dt}{1+t^2}$

$$I(\alpha)=\int_0^{\frac{\pi}{2}}\cos(\alpha\tan x)dx=\int_0^{\infty}\frac{\cos\alpha t}{1+t^2}dt=\frac{1}{\alpha}\left[\frac{\sin\alpha t}{1+t^2}\right]_0^{\infty}+\frac{1}{\alpha}\int_0^{\infty}\frac{2t\sin\alpha t}{(1+t^2)^2}dt$$

$=\dfrac{1}{\alpha}\displaystyle\int_0^{\infty}\frac{2t\sin\alpha t}{(1+t^2)^2}dt$, $\alpha I(\alpha)$에 대하여 파인만 적분 테크닉을 이용하면

$$\frac{d}{d\alpha}(\alpha I(\alpha))=I(\alpha)+\alpha I'(\alpha)=\int_0^{\infty}\frac{\partial}{\partial\alpha}\left(\frac{2t\sin\alpha t}{(1+t^2)^2}\right)dt=\int_0^{\infty}\frac{2t^2\cos\alpha t}{(1+t^2)^2}dt$$

$$=\int_0^{\infty}\frac{2(t^2+1-1)\cos\alpha t}{(1+t^2)^2}dt=2\int_0^{\infty}\frac{\cos\alpha t}{1+t^2}dt-\int_0^{\infty}\frac{2\cos\alpha t}{(1+t^2)^2}dt$$

$$=2I(\alpha)-\int_0^{\infty}\frac{2\cos\alpha t}{(1+t^2)^2}dt.$$

다시 양변을 $\alpha$에 대하여 미분하면

$$I'(\alpha)+\alpha I''(\alpha)+I'(\alpha)=2I'(\alpha)-\int_0^{\infty}\frac{\partial}{\partial\alpha}\frac{2\cos\alpha t}{(1+t^2)^2}dt=2I'(\alpha)+\int_0^{\infty}\frac{2t\sin\alpha t}{(1+t^2)^2}dt$$

$\alpha I''(\alpha)=\displaystyle\int_0^{\infty}\frac{2t\sin\alpha t}{(1+t^2)^2}dt=\alpha I(\alpha)$, $\quad I''(\alpha)=I(\alpha)$라는 미분방정식을 얻는다.

$I(\alpha)$의 이계도함수가 $I(\alpha)$와 동일해야 하므로 $I(\alpha)$는 지수함수 또는 삼각함수들의 합으로 구성되어 있어야 하고, 상수 $C_1$, $C_2$, $C_3$, $C_4$에 대하여
$I(\alpha)=C_1e^{f(\alpha)}+C_2e^{-f(\alpha)}$와 $I(\alpha)=C_3\sin g(\alpha)+C_4\cos g(\alpha)$의 형태를 모두 시험해보면 다음과 같은 해를 얻는다. (오일러공식 $e^{ix}=\cos x+i\sin x$를 적용하면 두 경우가 동일함을 알 수 있다.)

$$I(\alpha)=C_1e^{\alpha}+C_2e^{-\alpha}\quad(\alpha\geq 0)$$

이때 $I(\alpha)$는 $\cos(\alpha\tan x)$를 $0$부터 $\dfrac{\pi}{2}$까지 정적분한 값이므로 모든 $\alpha\geq 0$에 대하여 유계(bounded)이고, 따라서 $C_1=0$이다. 또한 $\alpha=0$을 대입하면 $I(0)=\dfrac{\pi}{2}=C_2$를 얻는다.

$$I(\alpha)=\int_0^{\frac{\pi}{2}}\cos(\alpha\tan x)dx=\frac{\pi}{2e^{\alpha}},$$

$$\therefore\int_0^{\frac{\pi}{2}}\cos(\tan x)dx=I(1)=\frac{\pi}{2e}\ ■$$

53. $I(\alpha):=\int_0^{\frac{\pi}{2}}\frac{\tan^{-1}(\alpha\tan x)}{\tan x}dx$

$$\frac{d}{d\alpha}I(\alpha)=\int_0^{\frac{\pi}{2}}\frac{\partial}{\partial\alpha}\frac{\tan^{-1}(\alpha\tan x)}{\tan x}dx=\int_0^{\frac{\pi}{2}}\frac{1}{1+\alpha^2\tan^2 x}dx$$

$$=\int_0^{\infty}\frac{1}{(1+\alpha^2t^2)(1+t^2)}dt\ \ (t=\tan x,\ dt=\sec^2 x dx)$$

$$=\frac{1}{\alpha^2-1}\int_0^{\infty}\left(\frac{\alpha^2}{1+\alpha^2t^2}-\frac{1}{1+t^2}\right)dt=\frac{1}{\alpha^2-1}\left[\alpha\tan^{-1}(\alpha t)-\tan^{-1}t\right]_0^{\infty}=\frac{\pi}{2(\alpha+1)}$$

$I(\alpha)=\frac{\pi}{2}\ln|\alpha+1|+C,\quad C=I(0)=0$에서 $I(\alpha)=\frac{\pi}{2}\ln|\alpha+1|$

$$\therefore\int_0^{\frac{\pi}{2}}\frac{x}{\tan x}dx=I(1)=\frac{\pi}{2}\ln 2\ ■$$

54. $I(\alpha):=\int_0^{\pi}\ln(1-2\alpha\cos x+\alpha^2)dx,\ \frac{d}{d\alpha}I(\alpha)=\int_0^{\pi}\frac{\partial}{\partial\alpha}\ln(1-2\alpha\cos x+\alpha^2)dx$

$$=\int_0^{\pi}\frac{2(\alpha-\cos x)}{1-2\alpha\cos x+\alpha^2}dx=\frac{1}{\alpha}\int_0^{\pi}\frac{2(\alpha^2-\alpha\cos x)}{1-2\alpha\cos x+\alpha^2}dx$$

$$=\frac{1}{\alpha}\int_0^{\pi}\left(\frac{2(\alpha^2-\alpha\cos x)}{1-2\alpha\cos x+\alpha^2}-1+1\right)dx$$

$$=\frac{1}{\alpha}\int_0^{\pi}\left(\frac{2(\alpha^2-\alpha\cos x)}{1-2\alpha\cos x+\alpha^2}-\frac{1-2\alpha\cos x+\alpha^2}{1-2\alpha\cos x+\alpha^2}+1\right)dx$$

$$=\frac{1}{\alpha}\int_0^{\pi}\left(1-\frac{1-\alpha^2}{1+\alpha^2-2\alpha\cos x}\right)dx=\frac{\pi}{\alpha}-\frac{1-\alpha^2}{\alpha(1+\alpha^2)}\int_0^{\pi}\left(\frac{1}{1-\frac{2\alpha}{1+\alpha^2}\cos x}\right)dx$$

$$= \frac{\pi}{\alpha} - \frac{1-\alpha^2}{\alpha(1+\alpha^2)} \int_0^\infty \frac{2}{(1+t^2)\left(1-\frac{2\alpha}{1+\alpha^2}\frac{1-t^2}{1+t^2}\right)} dt \quad \left(t = \tan\frac{x}{2},\ dt = \frac{1}{2}\sec^2\frac{x}{2}dx\right)$$

$$= \frac{\pi}{\alpha} - \frac{1-\alpha^2}{\alpha(1+\alpha^2)} \int_0^\infty \frac{2}{\left(1-\frac{2\alpha}{1+\alpha^2}\right)+\left(1+\frac{2\alpha}{1+\alpha^2}\right)t^2} dt$$

$$= \frac{\pi}{\alpha} - \frac{1-\alpha^2}{\alpha(1+\alpha^2)} \int_0^\infty \frac{2(1+\alpha^2)}{(1+\alpha^2-2\alpha)+(1+\alpha^2+2\alpha)t^2} dt$$

$$= \frac{\pi}{\alpha} - \frac{1-\alpha^2}{\alpha(1+\alpha^2)} \int_0^\infty \frac{2(1+\alpha^2)}{(1-\alpha)^2+(1+\alpha)^2 t^2} dt = \frac{\pi}{\alpha} - \frac{2(1+\alpha)}{\alpha(1-\alpha)} \int_0^\infty \frac{1}{1+\left(\frac{\alpha+1}{\alpha-1}\right)^2 t^2} dt$$

$$= \frac{\pi}{\alpha} - \frac{2(1+\alpha)}{\alpha(1-\alpha)} \left[\frac{\alpha-1}{\alpha+1}\tan^{-1}\left(\frac{\alpha+1}{\alpha-1}\right)t\right]_0^\infty = \frac{\pi}{\alpha} + \frac{\pi}{\alpha} = \frac{2\pi}{\alpha}$$

$$I(\alpha) = \int \frac{2\pi}{\alpha} d\alpha = 2\pi\ln|\alpha| + C,\ I(1) = \int_0^\pi \ln(2-2\cos x)dx = 0 = C \ (※)$$

$$I(\alpha) = \int_0^\pi \ln(1-2\alpha\cos x+\alpha^2)dx = 2\pi\ln|\alpha|$$

$$\therefore \int_0^\pi \ln(1-2e\cos x+e^2)dx = I(e) = 2\pi \ ■$$

$$※ \quad J = \int_0^\pi \ln(2-2\cos x)dx = \pi\ln 2 + \int_0^\pi \ln(1-\cos x)dx$$

$$= \pi\ln 2 + \int_0^\pi \ln(1+\cos x)dx \quad (x \mapsto \pi - x)$$

$$= \pi\ln 2 + \frac{1}{2}\int_0^\pi \ln(1-\cos^2 x)dx = \pi\ln 2 + \frac{1}{2}\int_0^\pi \ln(\sin^2 x)dx$$

$$= \pi\ln 2 + \int_0^\pi \ln(\sin x)dx = \pi\ln 2 + 2K$$

$$K = \frac{1}{2}\int_0^\pi \ln(\sin x)dx = \int_0^{\frac{\pi}{2}} \ln(\sin x)dx = \int_0^{\frac{\pi}{2}} \ln(\cos x)dx \quad \left(x \mapsto \frac{\pi}{2} - x\right)$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln(\sin x \cos x)dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\left(\frac{1}{2}\sin 2x\right)dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln(\sin 2x)dx - \frac{\pi}{4}\ln 2$$

$$= \frac{1}{4}\int_0^{\pi} \ln(\sin t)dt - \frac{\pi}{4}\ln 2 \ \ (t = 2x,\ dt = 2dx)$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln(\sin t)dt - \frac{\pi}{4}\ln 2 = \frac{1}{2}I - \frac{\pi}{4}\ln 2, \quad \frac{1}{2}I = -\frac{\pi}{4}\ln 2$$

$$\therefore K = \int_0^{\frac{\pi}{2}} \ln(\sin x)dx = -\frac{\pi}{2}\ln 2$$

$\therefore J = \pi \ln 2 + 2K = 0$ ■

55. $x = \tan t,\ dx = \sec^2 t dt$

$$I = \int_0^{\pi} \ln(1+\sin^2 t)dt = 2\int_0^{\frac{\pi}{2}} \ln(1+\sin^2 t)dt = 2\int_0^{\infty} \frac{\ln(2+x^2) - \ln(1+x^2)}{1+x^2}dx$$

$$I(\alpha) := \int_0^{\infty} \frac{\ln(\alpha+x^2) - \ln(1+x^2)}{1+x^2}dx,$$

$$\frac{d}{d\alpha}I(\alpha) = \int_0^{\infty} \frac{\partial}{\partial \alpha}\frac{\ln(\alpha+x^2) - \ln(1+x^2)}{1+x^2}dx = \int_0^{\infty} \frac{1}{(1+x^2)(\alpha+x^2)}dx$$

$$= \frac{1}{\alpha-1}\left(\int_0^{\infty}\frac{1}{x^2+1}dx - \int_0^{\infty}\frac{1}{x^2+\alpha}dx\right) = \frac{1}{\alpha-1}\left(\left[\tan^{-1}x\right]_0^{\infty} - \left[\frac{1}{\sqrt{\alpha}}\tan^{-1}\left(\frac{x}{\sqrt{\alpha}}\right)\right]_0^{\infty}\right)$$

$$= \frac{\pi}{2\sqrt{\alpha}(1+\sqrt{\alpha})}$$

한편 $I(1) = \int_0^{\infty} 0\,dx = 0$이므로

$$I = 2I(2) = 2\int_1^2 \frac{d}{d\alpha}I(\alpha)d\alpha = \pi\int_1^2 \frac{1}{\sqrt{\alpha}(1+\sqrt{\alpha})}d\alpha$$

$$= 2\pi\int_1^{\sqrt{2}} \frac{1}{1+u}du \ \ (\alpha = u^2,\ d\alpha = 2udu)$$

$$= 2\pi\left[\ln|1+u|\right]_1^{\sqrt{2}} = 2\pi\ln\left(\frac{1+\sqrt{2}}{2}\right) \blacksquare$$

56. $x = u+2,\ dx = du$

$$I = \int_0^4 \frac{\ln x}{\sqrt{4x-x^2}}dx = \int_{-2}^{2}\frac{\ln(u+2)}{\sqrt{4-u^2}}du = \int_0^2 \frac{\ln(u+2)}{\sqrt{4-u^2}}du + \int_{-2}^{0}\frac{\ln(u+2)}{\sqrt{4-u^2}}du$$

$$= \int_0^2 \frac{\ln(u+2)}{\sqrt{4-u^2}}du + \int_2^0 \frac{\ln(-u+2)}{\sqrt{4-u^2}}(-du)\ \ (u \mapsto -u)$$

$$= \int_0^2 \frac{\ln(4-u^2)}{\sqrt{4-u^2}}du$$

한편 $x = 4-u^2,\ dx = -2udu$의 치환을 하면

$$I = \int_0^4 \frac{\ln x}{\sqrt{4x-x^2}}dx = \int_2^0 \frac{\ln(4-u^2)}{u\sqrt{4-u^2}}\cdot(-2udu) = 2\int_0^2 \frac{\ln(4-u^2)}{\sqrt{4-u^2}}du$$

$$I = 2I,\ \ \therefore I = \int_0^4 \frac{\ln x}{\sqrt{4x-x^2}}dx = 0 \blacksquare$$

57. $x = \tan t,\ dx = \sec^2 t dt,\ \dfrac{dx}{1+x^2} = dt$

$$I = \int_0^\infty \frac{1}{(1+x^2)(1+x^\pi)}dx = \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^\pi t}dt$$

$$= \int_{\frac{\pi}{2}}^0 \frac{1}{1+\tan^\pi\left(\frac{\pi}{2}-t\right)}(-dt) = \int_0^{\frac{\pi}{2}}\frac{1}{1+\cot^\pi t}dt = \int_0^{\frac{\pi}{2}}\frac{\tan^\pi t}{1+\tan^\pi t}dt\ \ \left(t \mapsto \frac{\pi}{2}-t\right)$$

$$2I = \int_0^{\frac{\pi}{2}}\left(\frac{1}{1+\tan^\pi t}+\frac{\tan^\pi t}{1+\tan^\pi t}\right)dt = \frac{\pi}{2}$$

$$\therefore I = \int_0^\infty \frac{1}{(1+x^2)(1+x^\pi)}dx = \frac{\pi}{4} \blacksquare$$

58. sol 1) $I(v):=\int_0^\infty \frac{1}{(v+x)(\pi^2+(\ln x)^2)}dx$, $J(v):=\int_{-\infty}^\infty \frac{1}{(v+e^t)(\pi^2+t^2)}dt$

$x=e^t$, $dx=e^t dt$

$$I(v)=\int_{-\infty}^\infty \frac{e^t}{(v+e^t)(\pi^2+t^2)}dt=\int_{-\infty}^\infty \frac{(v+e^t)}{(v+e^t)(\pi^2+t^2)}dt-v\int_{-\infty}^\infty \frac{1}{(v+e^t)(\pi^2+t^2)}dt$$

$$=\left[\frac{1}{\pi}\tan^{-1}\left(\frac{x}{\pi}\right)\right]_{-\infty}^\infty - vJ(v)=1-vJ(v)$$

$t=-z$, $dt=-dz$

$$I(v)=\int_\infty^{-\infty}\frac{e^{-z}}{(v+e^{-z})(\pi^2+z^2)}(-dz)=\int_{-\infty}^\infty \frac{1}{(1+ve^z)(\pi^2+z^2)}dz=\frac{1}{v}J\left(\frac{1}{v}\right)$$

$$1-vJ(v)=I(v)=\frac{1}{v}J\left(\frac{1}{v}\right),\quad vJ(v)+\frac{1}{v}J\left(\frac{1}{v}\right)=1$$

$$v=1,\ \therefore J(1)=I(1)=\int_0^\infty \frac{1}{(1+x)(\pi^2+(\ln x)^2)}dx=\frac{1}{2}\ \blacksquare$$

sol 2) $I=\int_0^\infty \frac{1}{(1+x)(\pi^2+(\ln x)^2)}dx$

$$=\int_0^1 \frac{1}{(1+x)(\pi^2+(\ln x)^2)}dx+\int_1^\infty \frac{1}{(1+x)(\pi^2+(\ln x)^2)}dx$$

$$=\int_0^1 \frac{1}{(1+x)(\pi^2+(\ln x)^2)}dx+\int_1^0 \frac{1}{(1+1/x)(\pi^2+(\ln x)^2)}\cdot\left(-\frac{1}{x^2}\right)dx\ \ (x\mapsto\frac{1}{x})$$

$$=\int_0^1 \frac{1}{(1+x)(\pi^2+(\ln x)^2)}dx+\int_0^1 \frac{1}{x(1+x)(\pi^2+(\ln x)^2)}dx$$

$$=\int_0^1\left(\frac{1}{(1+x)(\pi^2+(\ln x)^2)}\cdot\frac{x+1}{x}\right)dx=\int_0^1 \frac{1}{x(\pi^2+(\ln x)^2)}dx$$

$$=\int_{-\infty}^0 \frac{1}{\pi^2+u^2}du\quad (u=\ln x,\ du=\frac{dx}{x})$$

$$= \left[ \frac{1}{\pi} \tan^{-1}\left(\frac{u}{\pi}\right)\right]_{-\infty}^{0} = \frac{1}{2} \ ■$$

59. $I = \int_0^\infty \frac{x-1}{\sqrt{2^x-1}\ln(2^x-1)}dx$

$$= \frac{1}{(\ln 2)^2}\int_0^\infty \frac{\ln(t^2+1)-\ln 2}{(t^2+1)\ln t}dt \ \ \left(t = \sqrt{2^x-1},\ dt = \frac{2^{x-1}\ln 2}{\sqrt{2^x-1}}dx\right)$$

$$= \frac{1}{(\ln 2)^2}\int_\infty^0 \frac{\ln(u^{-2}+1)-\ln 2}{\left(\frac{1}{u^2}+1\right)(-\ln u)}\left(-\frac{1}{u^2}\right)du = -\frac{1}{(\ln 2)^2}\int_0^\infty \frac{\ln(u^2+1)-\ln u^2-\ln 2}{(u^2+1)\ln u}du$$

$\left(u = \frac{1}{t},\ dt = -\frac{1}{u^2}du\right)$

$$= -\frac{1}{(\ln 2)^2}\int_0^\infty \frac{\ln(u^2+1)-\ln 2}{(u^2+1)\ln u}du + \frac{2}{(\ln 2)^2}\int_0^\infty \frac{1}{u^2+1}du$$

$$= -I + \frac{2}{(\ln 2)^2}\left[\tan^{-1}u\right]_0^\infty = -I + \frac{\pi}{(\ln 2)^2}$$

$$\therefore I = \int_0^\infty \frac{x-1}{\sqrt{2^x-1}\ln(2^x-1)}dx = \frac{\pi}{2(\ln 2)^2} \ ■$$

60. $t = \tan x,\ dt = \sec^2 x dx$

$$\int_0^{3\pi}\frac{1}{\sin^4 x+\cos^4 x}dx = 6\int_0^{\frac{\pi}{2}}\frac{\sec^4 x}{1+\tan^4 x}dx = 6\int_0^\infty \frac{1+t^2}{1+t^4}dt$$

$$= 6\int_0^\infty\left(\frac{1}{2(t^2+\sqrt{2}t+1)} - \frac{1}{2(-t^2+\sqrt{2}t-1)}\right)dt$$

$$= 3\int_0^\infty \frac{1}{t^2+\sqrt{2}t+1}dt + 3\int_0^\infty \frac{1}{t^2-\sqrt{2}t+1}dt$$

$$= 3\int_0^\infty \frac{1}{\left(t+\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}dt + 3\int_0^\infty \frac{1}{\left(t-\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}dt$$

$$= 3\left[\sqrt{2}\tan^{-1}(\sqrt{2}t+1)\right]_0^\infty + 3\left[\sqrt{2}\tan^{-1}(\sqrt{2}t-1)\right]_0^\infty = 3\sqrt{2}\pi \ ■$$

61. $\int_0^{\pi} \cos^4 x dx = \frac{1}{4}\int_0^{\pi} (1-\cos 2x)^2 dx = \frac{1}{4}\int_0^{\pi}\left(1-2\cos 2x+\frac{1+\cos 4x}{2}\right)dx$

$= \int_0^{\pi}\left(\frac{3}{8}-\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x\right)dx = \left[\frac{3}{8}x-\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x\right]_0^{\pi} = \frac{3}{8}\pi$ ■

62. $\int_0^{\frac{\pi}{2}} \sin^6 x \cos^3 x dx = \int_0^{\frac{\pi}{2}} \sin^6 x(1-\sin^2 x)\cos x dx$

$= \int_0^{\frac{\pi}{2}} \sin^6 x \cos x dx - \int_0^{\frac{\pi}{2}} \sin^8 x \cos x dx = \left[\frac{1}{7}\sin^7 x-\frac{1}{9}\sin^9 x\right]_0^{\frac{\pi}{2}} = \frac{2}{63}$ ■

63. $\int_0^{\frac{\pi}{4}} \sin^3 x \sec^2 x dx = \int_0^{\frac{\pi}{4}} \frac{\sin x(1-\cos^2 x)}{\cos^2 x}dx = \int_0^{\frac{\pi}{4}}(\tan x\sec x-\sin x)dx$

$= [\sec x+\cos x]_0^{\frac{\pi}{4}} = \frac{3\sqrt{2}}{2}-2$ ■

64. $x = 2\tan t,\ dx = 2\sec^2 t dt$

$\int_1^3 \frac{1}{x^2\sqrt{x^2+4}}dx = \int_{\alpha}^{\beta}\frac{\sec t}{4\tan^2 t}dt = \frac{1}{4}\int_{\alpha}^{\beta}\csc t\cot t dt = \left[-\frac{1}{4}\csc t\right]_{\alpha}^{\beta} = \left[-\frac{\sqrt{x^2+4}}{4x}\right]_1^3$

$= \frac{3\sqrt{5}-\sqrt{13}}{12}$ ■ $\left(\alpha = \tan^{-1}\left(\frac{1}{2}\right),\ \beta = \tan^{-1}\left(\frac{3}{2}\right)\right)$

65. $x+1 = \tan t,\ dx = \sec^2 t dt$

$\int_{\sqrt{3}-1}^{2\sqrt{2}-1}\frac{1}{(x+1)\sqrt{x^2+2x+2}}dx = \int_{\alpha}^{\beta}\frac{\sec^2 t}{\tan t\sec t}dt = \int_{\alpha}^{\beta}\csc t dt = [\ln|\csc t-\cot t|]_{\alpha}^{\beta}$

$= \left[\ln\left|\frac{\sqrt{x^2+2x+2}-1}{x+1}\right|\right]_{\sqrt{3}-1}^{2\sqrt{2}-1} = \ln\sqrt{\frac{3}{2}}$ ■ $\left(\alpha = \frac{\pi}{3},\ \beta = \tan^{-1}(2\sqrt{2})\right)$

66. $\int_0^{\tan^{-1}(\sqrt{6})}\frac{1}{1+\cos^2 x}dx = \int_0^{\tan^{-1}(\sqrt{6})}\sec^2 x\cdot\frac{1}{\tan^2 x+2}dx$

$$= \left[ \frac{1}{\sqrt{2}} \tan^{-1}\left( \frac{\tan x}{\sqrt{2}} \right) \right]_0^{\tan^{-1}(\sqrt{6})} = \frac{\sqrt{2}}{6}\pi \ ■$$

67. sol 1) $\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x} dx = \int_0^{\frac{\pi}{2}} \frac{1-\cos^2 x}{\sin^2 x} dx = \int_0^{\frac{\pi}{2}} (\csc^2 x - \csc x \cot x) dx$

$$= [\csc x - \cot x]_0^{\frac{\pi}{2}} = 1 - \lim_{x\to 0}(\csc x - \cot x) = 1 \ ■$$

sol 2) $\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x} dx = \int_0^{\frac{\pi}{2}} \frac{1}{2\cos^2 \frac{x}{2}} dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} \sec^2 \frac{x}{2} dx = \left[ \tan \frac{x}{2} \right]_0^{\frac{\pi}{2}} = 1$ ■

68. $t = e^x + 1, \ dt = e^x dx$

$$\int_{\ln(e-1)}^{\ln(e^3-1)} \frac{e^x \ln(e^x+1)}{e^x+1} dx = \int_e^{e^3} \frac{\ln t}{t} dt = \left[ \frac{1}{2}(\ln t)^2 \right]_e^{e^3} = 4 \ ■$$

69. $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{\sin x} dx = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \csc x dx = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\csc^2 x - \csc x \cot x}{\csc x - \cot x} dx$

$$= [\ln|\csc x - \cot x|]_{\pi/3}^{\pi/2} = \ln\sqrt{3} \ ■$$

70. $\int_0^{\frac{\pi}{4}} \frac{1}{1-3\cos^2 x} dx = \int_0^{\frac{\pi}{4}} \sec^2 x \cdot \frac{1}{\tan^2 x - 2} dx = \left[ \frac{1}{2\sqrt{2}} \ln \left| \frac{\tan x - \sqrt{2}}{\tan x + \sqrt{2}} \right| \right]_0^{\frac{\pi}{4}}$

$$= \frac{1}{2\sqrt{2}} \ln(3-2\sqrt{2}) = \frac{1}{\sqrt{2}} \ln(\sqrt{2}-1) \ ■$$

71. $\int_{-1}^0 \frac{x^3-x-2}{x^3-x^2+x-1} dx = \int_{-1}^0 \left( 1 - \frac{1}{x-1} + \frac{2x}{x^2+1} \right) dx = [x - \ln|x-1| + \ln(x^2+1)]_{-1}^0$

$$= 1 \ ■$$

72. $\int_3^4 \frac{x+4}{x^3+3x^2-10x} dx = \int_3^4 \frac{x+4}{x(x+5)(x-2)} dx = \int_3^4 \left( -\frac{2}{5x} - \frac{1}{35(x+5)} + \frac{3}{7(x-2)} \right) dx$

$$= \left[ -\frac{2}{5}\ln|x| - \frac{1}{35}\ln|x+5| + \frac{3}{7}\ln|x-2| \right]_3^4 = \frac{12}{35}\ln 3 - \frac{10}{35}\ln 2 = \frac{2}{35}\ln\left( \frac{729}{32} \right) \ ■$$

73. $\int_5^6 \frac{7x^3-13x^2-24x+24}{x^4-3x^3-10x^2+24x}dx = \int_5^6 \frac{7x^3-13x^2-24x+24}{x(x-2)(x+3)(x-4)}dx$

$= \int_5^6 \left(\frac{1}{x-2}+\frac{1}{x}+\frac{2}{x+3}+\frac{3}{x-4}\right)dx = [\ln|x-2|+\ln|x|+2\ln|x+3|+3\ln|x-4|]_5^6$

$= 4\ln3-\ln5 = \ln\left(\frac{81}{5}\right)$ ■

74. $t=\frac{1}{x}$, $dt = -\frac{1}{x^2}dx$

$\int_1^2 \frac{1}{x^2\sqrt{2x-x^2}}dx = \int_1^2 \frac{1}{x^3\sqrt{\frac{2}{x}-1}}dx = -\int_1^{\frac{1}{2}} \frac{t}{\sqrt{2t-1}}dt = \frac{1}{2}\int_{\frac{1}{2}}^1 \frac{2t-1+1}{\sqrt{2t-1}}dt$

$= \frac{1}{2}\int_{\frac{1}{2}}^1 \left(\sqrt{2t-1}+\frac{1}{\sqrt{2t-1}}\right)dt = \frac{1}{2}\left[\frac{1}{6}(2t-1)^{\frac{3}{2}}+\frac{1}{2}\sqrt{2t-1}\right]_{\frac{1}{2}}^1 = \frac{2}{3}$ ■

75. $t=x^2$, $dt=2xdx$

$\int_1^e \frac{x^4+81}{x(x^2+9)^2}dx = \frac{1}{2}\int_1^{e^2}\frac{t^2+81}{t(t+9)^2}dt = \frac{1}{2}\int_1^{e^2}\left(\frac{1}{t}-\frac{18}{(t+9)^2}\right)dt$

$= \left[\frac{1}{2}\ln|t|+\frac{9}{t+9}\right]_1^{e^2} = \frac{e^2+99}{10(e^2+9)}$ ■

76. $t=\sqrt{x}$, $dt=\frac{1}{2\sqrt{x}}dx$, $t=2\sin\theta$, $dt=2\cos\theta d\theta$, $\sin\theta=\frac{\sqrt{x}}{2}$

$\int_0^3 \sqrt{\frac{4-x}{x}}dx = \int_0^{\sqrt{3}} \frac{\sqrt{4-t^2}}{t} \cdot 2tdt = 2\int_0^{\sqrt{3}} \sqrt{4-t^2}dt = 2\int_0^{\frac{\pi}{3}} 4\cos^2\theta d\theta$

$= 4\int_0^{\frac{\pi}{3}}(1+\cos2\theta)d\theta = [4\theta+4\sin\theta\cos\theta]_0^{\frac{\pi}{3}} = \sqrt{3}+\frac{4}{3}\pi$ ■

77. $t=x^{\frac{3}{2}}$, $dt=\frac{3}{2}\sqrt{x}dx$

$$\int_0^{\frac{1}{\sqrt[3]{2}}} \sqrt{\frac{x}{1-x^3}}\,dx = \frac{2}{3}\int_0^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{1-t^2}}\,dt = \left[\frac{2}{3}\sin^{-1}t\right]_0^{\frac{1}{\sqrt{2}}} = \frac{\pi}{6} \blacksquare$$

78. $t = \sqrt{x}$, $dt = \dfrac{1}{2\sqrt{x}}dx$, $t = \sin u$, $dt = \cos u\,du$

$$\int_0^{\frac{3}{4}} \sqrt{x}\sqrt{1-x}\,dx = \int_0^{\frac{\sqrt{3}}{2}} t\sqrt{1-t^2} \cdot 2t\,dt = 2\int_0^{\frac{\sqrt{3}}{2}} t^2\sqrt{1-t^2}\,dt$$

$$= 2\int_0^{\frac{\pi}{3}} \sin^2 u \cos u \cdot \cos u\,du = \frac{1}{2}\int_0^{\frac{\pi}{3}} \sin^2 2u\,du = \frac{1}{4}\int_0^{\frac{\pi}{3}} (1-\cos 4u)\,du$$

$$= \left[\frac{1}{4}u - \frac{1}{16}\sin 4u\right]_0^{\frac{\pi}{3}} = \frac{\sqrt{3}}{32} + \frac{\pi}{12} \blacksquare$$

79. $t = \tan x$, $dt = \sec^2 x\,dx$

$$\int_0^{\tan^{-1}(\sqrt[4]{3})} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x}\,dx = \int_0^{\tan^{-1}(\sqrt[4]{3})} \frac{\tan x \sec^2 x}{1+\tan^4 x}\,dx = \int_0^{\sqrt[4]{3}} \frac{t}{1+t^4}\,dt$$

$$= \left[\frac{1}{2}\tan^{-1}(t^2)\right]_0^{\sqrt[4]{3}} = \frac{\pi}{6} \blacksquare$$

80. $$\int_0^{\frac{\pi}{4}} \tan^4 x \sec^4 x\,dx = \int_0^{\frac{\pi}{4}} \tan^4 x(\tan^2 x + 1)\sec^2 x\,dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^6 x \sec^2 x\,dx + \int_0^{\frac{\pi}{4}} \tan^4 x \sec^2 x\,dx = \left[\frac{1}{7}\tan^7 x + \frac{1}{5}\tan^5 x\right]_0^{\frac{\pi}{4}} = \frac{12}{35} \blacksquare$$

81. $t = \sqrt{x}$, $dt = \dfrac{1}{2\sqrt{x}}dx$

$$\int_2^4 \frac{x\sqrt{x}-1}{x-\sqrt{x}}\,dx = \int_{\sqrt{2}}^2 \frac{t^3-1}{t^2-t} \cdot 2t\,dt = 2\int_{\sqrt{2}}^2 (t^2+t+1)\,dt = \left[\frac{2}{3}t^3 + t^2 + 2t\right]_{\sqrt{2}}^2$$

$$= \frac{34-10\sqrt{2}}{3} \blacksquare$$

82. $t=e^x$, $dt=e^x dx$, $u=\sqrt{t+1}$, $du=\dfrac{1}{2\sqrt{t+1}}dt$

$$\int_0^{\ln 3}\frac{1}{\sqrt{1+e^x}}dx=\int_1^3\frac{1}{t\sqrt{t+1}}dt=\int_{\sqrt{2}}^2\frac{2}{u^2-1}du=\left[\ln\left|\frac{u-1}{u+1}\right|\right]_{\sqrt{2}}^2$$

$$=-\ln(9-6\sqrt{2})\ \blacksquare$$

83. $\displaystyle\int_1^2\frac{x^2+2x-1}{2x^3+3x^2-2x}dx=\int_1^2\frac{x^2+2x-1}{x(x+2)(2x-1)}dx=\int_1^2\left(-\frac{1}{10(x+2)}+\frac{1}{5(2x-1)}+\frac{1}{2x}\right)dx$

$$=\left[-\frac{1}{10}\ln|x+2|+\frac{1}{10}\ln|2x-1|+\frac{1}{2}\ln|x|\right]_1^2=\frac{3}{10}\ln 2+\frac{1}{5}\ln 3=\frac{1}{10}\ln 72\ \blacksquare$$

84. $\displaystyle I=\int_0^{\frac{\pi}{3}}13e^{2x}\cos 3x dx=\left[\frac{13}{2}e^{2x}\cos 3x\right]_0^{\frac{\pi}{3}}+\frac{3}{2}\int_0^{\frac{\pi}{3}}13e^{2x}\sin 3x dx$

$$=-\frac{13}{2}(e^{2\pi/3}+1)+\frac{39}{2}\left\{\left[\frac{1}{2}e^{2x}\sin 3x\right]_0^{\frac{\pi}{3}}-\frac{1}{2}\int_0^{\frac{\pi}{3}}3e^{2x}\cos 3x dx\right\}$$

$$=-\frac{13}{2}(e^{2\pi/3}+1)-\frac{9}{4}I$$

$$\frac{13}{4}I=-\frac{13}{2}(e^{2\pi/3}+1)$$

$$\therefore I=\int_0^{\frac{\pi}{3}}13e^{2x}\cos 3x dx=-2(e^{2\pi/3}+1)\ \blacksquare$$

85. $\displaystyle\int_0^{\pi}e^{-x}\sin^2 2x dx=\frac{1}{2}\int_0^{\pi}e^{-x}(1-\cos 4x)dx=\frac{1}{2}[-e^{-x}]_0^{\pi}-\frac{1}{2}\int_0^{\pi}e^{-x}\cos 4x dx$

$$=\frac{1-e^{-\pi}}{2}-\frac{1}{2}\left(\left[\frac{1}{4}e^{-x}\sin 4x\right]_0^{\pi}+\frac{1}{4}\int_0^{\pi}e^{-x}\sin 4x dx\right)$$

$$=\frac{1-e^{-\pi}}{2}-\frac{1}{8}\left(\left[-\frac{1}{4}e^{-x}\cos 4x\right]_0^{\pi}-\frac{1}{4}\int_0^{\pi}e^{-x}\cos 4x dx\right)$$

$$=\frac{15(1-e^{-\pi})}{32}+\frac{1}{32}\int_0^{\pi}e^{-x}\cos 4x dx$$

$$\frac{17}{32}\int_0^{\pi} e^{-x}\cos 4x dx = \frac{1-e^{-\pi}}{32}, \quad \int_0^{\pi} e^{-x}\cos 4x dx = \frac{1-e^{-\pi}}{17}$$

$$\therefore \int_0^{\pi} e^{-x}\sin^2 2x dx = \frac{1-e^{-\pi}}{2} - \frac{1}{2}\int_0^{\pi} e^{-x}\cos 4x dx = \frac{8}{17}(1-e^{-\pi}) \blacksquare$$

86. sol 1) $t^2 = \tan x$, $2tdt = \sec^2 x dx$, $dx = \dfrac{2t}{t^4+1}dt$

$$\int_0^{\frac{\pi}{4}} \sqrt{\tan x}\, dx = \int_0^1 \frac{2t^2}{t^4+1}dt = \int_0^1 \frac{2}{t^2+\frac{1}{t^2}}dt = \int_0^1 \frac{\left(1+\frac{1}{t^2}\right)+\left(1-\frac{1}{t^2}\right)}{t^2+\frac{1}{t^2}}dt$$

$$= \int_0^1 \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}dt + \int_0^1 \frac{1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}}dt = \int_0^1 \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+(\sqrt{2})^2}dt + \int_0^1 \frac{1-\frac{1}{t^2}}{\left(t+\frac{1}{t}\right)^2-(\sqrt{2})^2}dt$$

$$= \left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t-1/t}{\sqrt{2}}\right) + \frac{1}{2\sqrt{2}}\ln\left|\frac{t+1/t-\sqrt{2}}{t+1/t+\sqrt{2}}\right|\right]_0^1 = \frac{\pi}{2\sqrt{2}} + \frac{1}{\sqrt{2}}\ln(\sqrt{2}-1) \blacksquare$$

sol 2) $I := \displaystyle\int_0^{\frac{\pi}{4}} \sqrt{\tan x}\, dx$, $J := \displaystyle\int_0^{\frac{\pi}{4}} \sqrt{\cot x}\, dx$

$$I+J = \int_0^{\frac{\pi}{4}} (\sqrt{\tan x}+\sqrt{\cot x})dx = \sqrt{2}\int_0^{\frac{\pi}{4}} \frac{\sin x+\cos x}{\sqrt{\sin 2x}}dx$$

$$= \sqrt{2}\int_0^{\frac{\pi}{4}} \frac{(\sin x-\cos x)'}{\sqrt{1-(\sin x-\cos x)^2}}dx = \sqrt{2}\left[\sin^{-1}(\sin x-\cos x)\right]_0^{\frac{\pi}{4}} = \frac{\pi}{\sqrt{2}} \quad \cdots \quad [1]$$

$$I-J = \int_0^{\frac{\pi}{4}} (\sqrt{\tan x}-\sqrt{\cot x})dx = \sqrt{2}\int_0^{\frac{\pi}{4}} \frac{\sin x-\cos x}{\sqrt{\sin 2x}}dx$$

$$= -\sqrt{2}\int_0^{\frac{\pi}{4}} \frac{(\sin x+\cos x)'}{\sqrt{(\sin x+\cos x)^2-1}}dx$$

$$= \left[-\sqrt{2}\ln\left|(\sin x+\cos x)+\sqrt{(\sin x+\cos x)^2-1}\right|\right]_0^{\frac{\pi}{4}}$$

$$= -\sqrt{2}\ln(\sqrt{2}+1) = \sqrt{2}\ln(\sqrt{2}-1) \ \cdots \ [2]$$

$$\frac{[1]+[2]}{2} \ : \ \therefore I = \int_0^{\frac{\pi}{4}} \sqrt{\tan x}\, dx = \frac{\pi}{2\sqrt{2}} + \frac{1}{\sqrt{2}}\ln(\sqrt{2}-1) \ \blacksquare$$

87. $t = \tan x$, $dt = \sec^2 x dx$, $u = \dfrac{t}{1-t}$, $du = \dfrac{1}{(1-t)^2}dt$, $u = y^2$, $du = 2ydy$

$$I = \int_0^{\frac{\pi}{4}} \sqrt{\tan x}\sqrt{1-\tan x}\, dx = \int_0^1 \frac{\sqrt{t(1-t)}}{1+t^2}dt = \int_0^{\infty} \frac{\sqrt{u}}{(1+u)(1+2u+2u^2)}du$$

$$= \int_0^{\infty} \frac{2y^2}{(1+y^2)(1+2y^2+2y^4)}dy = 2\int_0^{\infty}\left(\frac{2y^2}{1+2y^2+2y^4} + \frac{1}{1+2y^2+2y^4} - \frac{1}{1+y^2}\right)dy$$

$$= \int_{-\infty}^{\infty}\left(\frac{2y^2}{1+2y^2+2y^4} + \frac{1}{1+2y^2+2y^4} - \frac{1}{1+y^2}\right)dy = I_1 + I_2 - \left[\tan^{-1}y\right]_{-\infty}^{\infty} = I_1 + I_2 - \pi$$

$$I_1 = \int_{-\infty}^{\infty} \frac{2y^2}{1+2y^2+2y^4}dy = \int_{-\infty}^{\infty} \frac{1}{y^2 + \frac{1}{2y^2} + 1}dy = \int_{-\infty}^{\infty} \frac{1}{\left(y - \frac{1}{\sqrt{2}y}\right)^2 + 1 + \sqrt{2}}dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{y^2 + 1 + \sqrt{2}}dy \ (\because \text{ Glasser's Master Theorem})$$

$$= \left[\frac{1}{\sqrt{1+\sqrt{2}}}\tan^{-1}\left(\frac{y}{\sqrt{1+\sqrt{2}}}\right)\right]_{-\infty}^{\infty} = \frac{\pi}{\sqrt{1+\sqrt{2}}}$$

$$I_2 = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{y^4 + y^2 + \frac{1}{2}}dy = \int_0^{\infty} \frac{1}{y^2 + \frac{1}{2y^2} + 1} \cdot \frac{dy}{y^2} = \int_0^{\infty} \frac{1}{\frac{y^2}{2} + \frac{1}{y^2} + 1}dy \ \left(y \mapsto \frac{1}{y}\right)$$

$$= \int_{-\infty}^{\infty} \frac{1}{y^2 + \frac{2}{y^2} + 2}dy = \int_{-\infty}^{\infty} \frac{1}{\left(y - \frac{\sqrt{2}}{y}\right)^2 + 2 + 2\sqrt{2}}dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{y^2 + 2 + 2\sqrt{2}}dy \ (\because \text{ Glasser's Master Theorem})$$

$$= \left[\frac{1}{\sqrt{2+2\sqrt{2}}}\tan^{-1}\left(\frac{y}{\sqrt{2+2\sqrt{2}}}\right)\right]_{-\infty}^{\infty} = \frac{\pi}{\sqrt{2+2\sqrt{2}}}$$

$$\therefore I = \int_0^{\frac{\pi}{4}} \sqrt{\tan x}\,\sqrt{1-\tan x}\,dx = I_1 + I_2 - \pi = \left(\sqrt{\frac{1+\sqrt{2}}{2}} - 1\right)\pi \ \blacksquare$$

88. $I_{m,\,n} := \displaystyle\int_1^{\infty} \frac{(\ln x)^m}{x^n}dx = \int_1^{\infty}\frac{(\ln x)^m}{x}\cdot\frac{1}{x^{n-1}}dx$

$$= \left[\frac{(\ln x)^{m+1}}{(m+1)x^{n-1}}\right]_1^{\infty} - \frac{1-n}{m+1}\int_1^{\infty}\frac{(\ln x)^{m+1}}{x^n}dx = \frac{n-1}{m+1}\int_1^{\infty}\frac{(\ln x)^{m+1}}{x^n}dx$$

$= \dfrac{n-1}{m+1}I_{m+1,\,n}$,　$I_{m,\,n} = \dfrac{m}{n-1}I_{m-1,\,n}$을 얻는다. 한편

$I_{0,\,n} = \displaystyle\int_1^{\infty}\frac{1}{x^n}dx = \frac{1}{n-1}$이므로

$I_{m,\,n} = \dfrac{m!}{(n-1)^{m+1}}$, $m,\ n \in \mathbb{N}_0$.

$$\therefore \int_1^{\infty}\frac{(\ln x)^{627}}{x^{2022}}dx = I_{627,\,2022} = \frac{627!}{2021^{628}} \ \blacksquare$$

89. $x = \cot t$, $\sin(\cot^{-1}x) = \sin t = \dfrac{x}{\sqrt{1+x^2}}$, $x = \tan u$, $\cos(\tan^{-1}x) = \cos u = \dfrac{1}{\sqrt{1+x^2}}$

$$\int_0^{\sqrt{6}} \cos^2(\tan^{-1}(\sin(\cot^{-1}x)))dx = \int_0^{\sqrt{6}}\cos^2\left(\tan^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\right)dx$$

$$= \int_0^{\sqrt{6}}\frac{1+x^2}{2+x^2}dx = \int_0^{\sqrt{6}}\left(1-\frac{1}{2+x^2}\right)dx = \left[x - \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_0^{\sqrt{6}} = \sqrt{6} - \frac{\pi}{3\sqrt{2}} \ \blacksquare$$

90. $t = \tan x$, $dt = \sec^2 x\,dx$, $u = t + \sqrt{1+t^2}$, $t = \dfrac{1}{2}\left(u - \dfrac{1}{u}\right)$, $dt = \dfrac{1}{2}\left(1 + \dfrac{1}{u^2}\right)du$

$$\int_0^{\frac{\pi}{6}}\frac{\sec^2 x}{(\sec x + \tan x)^{5/2}}dx = \int_0^{\frac{1}{\sqrt{3}}}\frac{1}{(t+\sqrt{1+t^2})^{5/2}}dt = \frac{1}{2}\int_1^{\sqrt{3}}(u^{-5/2} + u^{-9/2})du$$

$$= \left[ -\frac{1}{3}u^{-3/2} - \frac{1}{7}u^{-7/2} \right]_1^{\sqrt{3}} = -\frac{8}{63}\sqrt[4]{3} + \frac{10}{21}$$ ■

91. 제 1종 오일러 치환 : $\sqrt{x^2+4x-4} = x+t$, $x = \dfrac{t^2+4}{4-2t}$, $dx = \dfrac{-2t^2+8t+8}{(4-2t)^2}dt$

$$\int_1^2 \frac{1}{x\sqrt{x^2+4x-4}}dx = \int_0^{2\sqrt{2}-2} \frac{4-2t}{t^2+4} \cdot \frac{4-2t}{-t^2+4t+4} \cdot \frac{-2t^2+8t+8}{(4-2t)^2}dt$$

$$= 2\int_0^{2\sqrt{2}-2} \frac{1}{t^2+4}dt = \left[ \tan^{-1}\left(\frac{t}{2}\right) \right]_0^{2\sqrt{2}-2} = \tan^{-1}(\sqrt{2}-1) = \frac{\pi}{8}$$ ■

92. 제 2종 오일러 치환 : $\sqrt{-x^2+x+2} = xt + \sqrt{2}$, $x = \dfrac{1-2\sqrt{2}t}{t^2+1}$, $dx = \dfrac{2\sqrt{2}t^2-2t-2\sqrt{2}}{(t^2+1)^2}dt$

$$\int_1^{\frac{25}{73}} \frac{1}{x\sqrt{-x^2+x+2}}dx$$

$$= \int_0^{\frac{4\sqrt{2}}{25}} \frac{t^2+1}{1-2\sqrt{2}t} \cdot \frac{t^2+1}{-\sqrt{2}t^2+t+\sqrt{2}} \cdot \frac{2\sqrt{2}t^2-2t-2\sqrt{2}}{(t^2+1)^2}dt$$

$$= \int_0^{\frac{4\sqrt{2}}{25}} \frac{-2}{-2\sqrt{2}t+1}dt = \frac{1}{\sqrt{2}}\int_0^{\frac{4\sqrt{2}}{25}} \frac{-2\sqrt{2}}{-2\sqrt{2}t+1}dt = \left[ \frac{1}{\sqrt{2}}\ln\left|2\sqrt{2}t-1\right| \right]_0^{\frac{4\sqrt{2}}{25}}$$

$$= \sqrt{2}\ln\frac{3}{5}$$ ■

93. 제 3종 오일러 치환 : $\sqrt{-(x-2)(x-1)} = (x-2)t$, $x = \dfrac{-2t^2-1}{-t^2-1}$, $dx = \dfrac{2t}{(-t^2-1)^2}dt$

$$\int_1^2 \frac{x^2}{\sqrt{-x^2+3x-2}}dx = \int_0^{-\infty} \left( \frac{-2t^2-1}{-t^2-1} \right)^2 \cdot \frac{-t^2-1}{t} \cdot \frac{2t}{(-t^2-1)^2}dt$$

$$= -2\int_0^{-\infty} \frac{(2t^2+1)^2}{(t^2+1)^3}dt = -2\int_0^{-\infty} \left( \frac{4}{t^2+1} - \frac{4}{(t^2+1)^2} + \frac{1}{(t^2+1)^3} \right)dt$$

$$= \left[ -8\tan^{-1}t + 8I(t) - 2J(t) \right]_0^{-\infty} = \left[ -\frac{19}{4}\tan^{-1}t + \frac{t(13t^2+11)}{4(t^2+1)^2} \right]_0^{-\infty} = \frac{19}{8}\pi$$ ■

※ $t=\tan u$, $dt=\sec^2 u du$, $I(t)=\int_0^{-\infty}\frac{1}{(t^2+1)^2}dt=\int_0^{-\frac{\pi}{2}}\cos^2 u du$

$$=\frac{1}{2}\int_0^{-\frac{\pi}{2}}(1+\cos 2u)du=\left[\frac{1}{2}u+\frac{1}{4}\sin 2u\right]_0^{-\frac{\pi}{2}}=\left[\frac{1}{2}\tan^{-1}t+\frac{t}{2(1+t^2)}\right]_0^{-\infty}$$

※※ $t=\tan u$, $dt=\sec^2 u du$, $J(t)=\int_0^{-\infty}\frac{1}{(t^2+1)^3}dt=\int_0^{-\frac{\pi}{2}}\cos^4 u du$

$$=\frac{1}{4}\int_0^{-\frac{\pi}{2}}(1+\cos 2u)^2du=\frac{1}{4}\int_0^{-\frac{\pi}{2}}(1+2\cos 2u+\cos^2 2u)du$$

$$=\frac{1}{4}\int_0^{-\frac{\pi}{2}}\left(1+2\cos 2u+\frac{1+\cos 4u}{2}\right)du=\frac{1}{8}\int_0^{-\frac{\pi}{2}}(3+4\cos 2u+\cos 4u)du$$

$$=\left[\frac{3}{8}u+\frac{1}{4}\sin 2u+\frac{1}{32}\sin 4u\right]_0^{-\frac{\pi}{2}}=\left[\frac{3}{8}\tan^{-1}t+\frac{t(3t^2+5)}{8(t^2+1)^2}\right]_0^{-\infty}$$

94. $t=\cos\theta$, $dt=-\sin\theta d\theta$, $u=\frac{t-1}{t+1}$, $t=\frac{1+u}{1-u}$, $dt=\frac{2}{(1-u)^2}du$

$$\int_0^{\frac{\pi}{4}}\frac{\sin^3(\theta/2)}{\cos(\theta/2)\cdot\sqrt{\cos^3\theta+\cos^2\theta+\cos\theta}}d\theta$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{4}}\frac{(1-\cos\theta)\sin\theta}{(1+\cos\theta)\sqrt{\cos^3\theta+\cos^2\theta+\cos\theta}}d\theta=\frac{1}{2}\int_1^{\frac{1}{\sqrt{2}}}\frac{t-1}{(t+1)\sqrt{t^3+t^2+t}}dt$$

$$=\frac{1}{2}\int_0^{2\sqrt{2}-3}u\cdot\frac{\sqrt{(1-u)^3}}{\sqrt{(1+u)^3+(1+u)^2(1-u)+(1+u)(1-u)^2}}\cdot\frac{2}{(1-u)^2}du$$

$$=\int_0^{2\sqrt{2}-3}\frac{u}{\sqrt{3-2u^2-u^4}}du=\frac{1}{2}\int_0^{2\sqrt{2}-3}\frac{2u}{\sqrt{4-(u^2+1)^2}}du$$

$$=\left[\frac{1}{2}\sin^{-1}\left(\frac{u^2+1}{2}\right)\right]_0^{2\sqrt{2}-3}=\frac{1}{2}\sin^{-1}(9-6\sqrt{2})-\frac{\pi}{12}$$ ■

95. $\int_0^{\frac{\pi}{6}}\frac{\tan^4\theta}{1-\tan^2\theta}d\theta=\int_0^{\frac{\pi}{6}}\frac{\tan^4\theta-1}{1-\tan^2\theta}d\theta+\int_0^{\frac{\pi}{6}}\frac{1}{1-\tan^2\theta}d\theta$

$$= -\int_0^{\frac{\pi}{6}} (\tan^2\theta + 1)d\theta + \int_0^{\frac{\pi}{6}} \frac{\cos^2\theta}{\cos^2\theta - \sin^2\theta} d\theta = -\int_0^{\frac{\pi}{6}} \sec^2\theta d\theta + \int_0^{\frac{\pi}{6}} \frac{1+\cos 2\theta}{2\cos 2\theta} d\theta$$

$$= [-\tan\theta]_0^{\frac{\pi}{6}} + \int_0^{\frac{\pi}{6}} \left(\frac{1}{2}\sec 2\theta + \frac{1}{2}\right)d\theta = \left[-\tan\theta + \frac{1}{4}\ln|\sec 2\theta + \tan 2\theta| + \frac{1}{2}\theta\right]_0^{\frac{\pi}{6}}$$

$$= -\frac{1}{\sqrt{3}} + \frac{\pi}{12} + \frac{1}{4}\ln(2+\sqrt{3}) \ \blacksquare$$

96. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}dx = \int_{-\infty}^{\infty} \frac{1}{\left(x - \frac{1}{x}\right)^2 + 2} dx = \int_{-\infty}^{\infty} \frac{1}{x^2+2}dx$ ( $\because$ Glasser's Master Theorem)

$$= \left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_{-\infty}^{\infty} = \frac{\pi}{\sqrt{2}} \ \blacksquare$$

97. $\displaystyle\int_0^{\infty} \exp\left(-x^2 - \frac{1}{x^2}\right)dx = \int_0^{\infty} \exp\left[-\left(x - \frac{1}{x}\right)^2 - 2\right]dx = \frac{1}{2e^2}\int_{-\infty}^{\infty} \exp\left[-\left(x - \frac{1}{x}\right)^2\right]dx$

$= \dfrac{1}{2e^2}\displaystyle\int_{-\infty}^{\infty} e^{-u^2}du = \dfrac{\sqrt{\pi}}{2e^2} \ \blacksquare$ ( $\because$ Glasser's Master Theorem)

98. $\displaystyle\int_0^1 \frac{1-x^{99}}{(1+x)(1+x^{100})}dx = \int_0^1 \frac{(1+x^{100}) - x^{99}(1+x)}{(1+x)(1+x^{100})}dx = \int_0^1 \left(\frac{1}{1+x} - \frac{x^{99}}{1+x^{100}}\right)dx$

$$= \left[\ln|1+x| - \frac{1}{100}\ln|1+x^{100}|\right]_0^1 = \frac{99}{100}\ln 2 \ \blacksquare$$

99. $\displaystyle\int_{-\infty}^{\infty} \exp\left(-\frac{(x^2-13x-1)^2}{611x^2}\right)dx = \int_{-\infty}^{\infty} \exp\left(-\frac{(x - x^{-1} - 13)^2}{611}\right)dx$

$= \displaystyle\int_{-\infty}^{\infty} \exp\left(-\frac{(x-13)^2}{611}\right)dx$ ( $\because$ Glasser's Master Theorem)

$= \displaystyle\int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{611}\right)dx$ $(x \mapsto x+13,\ dx \mapsto dx)$

$= \displaystyle\int_{-\infty}^{\infty} \exp(-u^2) \cdot \sqrt{611}\,du$ $\left(u = \dfrac{x}{\sqrt{611}},\ dx = \sqrt{611}\,du\right)$

$= \sqrt{611\pi}$ ■

100. $t = \tan x$, $dt = \sec^2 x dx$

$$\int_0^{\frac{\pi}{2}} \sqrt[n]{\tan x}\, dx = \int_0^{\infty} \frac{t^{\frac{1}{n}}}{1+t^2} dt, \quad \int_0^{\infty} \frac{x^a}{1+x^2} dx = \frac{\pi}{2} \sec\left(\frac{a\pi}{2}\right)$$ 임을 이용하자.

sol 1) $t = e^x$, $dt = e^x dx$

$$\int_0^{\infty} \frac{x^a}{1+x^2} dx = \int_{-\infty}^{\infty} \frac{e^{(a+1)t}}{1+e^{2t}} dt = \int_{-\infty}^{0} \frac{e^{(a+1)t}}{1+e^{2t}} dt + \int_0^{\infty} \frac{e^{(a-1)t}}{1+e^{-2t}} dt$$

한편 $\dfrac{e^{(a+1)t}}{1-(-e^{2t})}$, $\dfrac{e^{(a-1)t}}{1-(-e^{-2t})}$는 공비가 각각 $(-e^{2t})$, $(-e^{-2t})$이고 초항이 각각 $e^{(a+1)t}$, $e^{(a-1)t}$인 등비급수의 합이므로 이를 역으로 전개할 수 있다.

$$\int_{-\infty}^{0} \frac{e^{(a+1)t}}{1+e^{2t}} dt + \int_0^{\infty} \frac{e^{(a-1)t}}{1+e^{-2t}} dt$$

$$= \int_{-\infty}^{0} \sum_{n=0}^{\infty} e^{(a+1)t} \cdot (-e^{2t})^n dt + \int_0^{\infty} \sum_{n=0}^{\infty} e^{(a-1)t} \cdot (-e^{-2t})^n dt$$

$$= \sum_{n=0}^{\infty} (-1)^n \left( \int_{-\infty}^{0} e^{(2n+1+a)t} dt + \int_0^{\infty} e^{-(2n+1-a)} dt \right)$$

$$= \sum_{n=0}^{\infty} (-1)^n \left( \frac{1}{2n+1+a} + \frac{1}{2n+1-a} \right) = 2\sum_{n=0}^{\infty} (-1)^n \left( \frac{2n+1}{(2n+1)^2 - a^2} \right) \quad \cdots \quad [\#]$$

한편 $\sec(z)$의 Mittag-Leffler 전개식은

$$\sec(z) = \pi \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{(\pi/2)^2 (2n+1)^2 - z^2}$$

이므로 (참고 : https://cdlbb.github.io/WandW/CMA07-3-FactorTheoremMN.html#7.4theexpansionofaclassoffunctionsinrationalfractions.)

$z = \dfrac{a\pi}{2}$를 대입하고 $\dfrac{\pi}{2}$를 곱하면

$$\frac{\pi}{2}\sec\left(\frac{a\pi}{2}\right)=\frac{\pi^2}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)}{(\pi/2)^2(2n+1)^2-(a\pi/2)^2}=2\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)}{(2n+1)^2-a^2}$$

을 얻고 이는 [#]의 전개식과 일치한다.

$$\therefore \int_0^{\infty}\frac{x^a}{1+x^2}dx=\frac{\pi}{2}\sec\left(\frac{a\pi}{2}\right) \blacksquare$$

sol 2) $t=x^2,\ dt=2xdx$

$$\int_0^{\infty}\frac{x^a}{1+x^2}dx=\frac{1}{2}\int_0^{\infty}\frac{t^{(a-1)/2}}{1+t}dt=\frac{1}{2}B\left(\frac{a+1}{2},\,1-\frac{a+1}{2}\right)$$

$$=\frac{1}{2}\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(1-\frac{a+1}{2}\right)=\frac{\pi}{2}\csc\left(\frac{(a+1)\pi}{2}\right)=\frac{\pi}{2}\sec\left(\frac{a\pi}{2}\right) \blacksquare$$

※ 베타함수 $B(a,\,b)$는 다음과 같이 정의되는 이변수 함수이다.

$$B(a,\,b):=\int_0^{\infty}\frac{z^{a-1}}{1+z^{a+b}}dz$$

또한 이는 다음과 같이 감마함수의 비로 표현할 수 있다.

$$B(a,\,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

감마함수 $\Gamma(z)$는 다음과 같이 정의되며, 이는 계승 함수($n!$)의 정의역을 실수부가 양수인 복소수 전체로 확장(음의 정수는 제외)한 것이다. (29번 문제의 1번 솔루션은 이 감마함수의 정의를 이용한 것으로, 자연수 $n$에 대하여 $\Gamma(n)=(n-1)!$이다.) 감마함수는 계승 함수를 해석적으로 확장한 것이므로 팩토리얼의 성질이 그대로 성립하여, $\Gamma(x+1)=x\Gamma(x)$이고 $\Gamma(1)=1$이다.)

$$\Gamma(z):=\int_0^{\infty}t^{z-1}e^{-t}dt\ \ (Re(z)>0)$$

감마함수에 대하여 다음과 같은 오일러 반사 공식이 성립하며, 상술한 2번 솔루션의 마지막 단계에서 이 공식이 사용되었다.

$$\Gamma(1-z)\Gamma(z)=\frac{\pi}{\sin(\pi z)}$$

위 공식은 Residue Theorem을 사용하여 직접 복소적분을 하거나 감마함수의 정의식을 대입하여 증명할 수 있으며, 본문에서는 복소적분을 사용하지 않는다는 철학을 유지하기 위해 다른 증명법을 제시한다. 복소적분을 포함한 다른 증명법에 관심이 있는 독자는 아래 링크의 증명들을 참고하기 바란다.

https://math.stackexchange.com/questions/714482/prove-that-gammap-times-gamma1-p-frac-pi-sin-p-pi-forall-p-in?noredirect=1&lq=1

pf) 감마함수의 바이어슈트라스 무한곱 꼴을 생각하자. (감마함수는 실제로 여러 가지 형태로 정의되고 이들은 모두 동치이다. 상술한 정의는 적분 형태이다.)

$$\frac{1}{\Gamma(p)} = pe^{\gamma p}\prod_{n=1}^{\infty}\left(1+\frac{p}{n}\right)e^{-\frac{p}{n}}$$

$$\frac{1}{\Gamma(p)}\cdot\frac{1}{\Gamma(-p)} = pe^{\gamma p}\prod_{n=1}^{\infty}\left(1+\frac{p}{n}\right)e^{-\frac{p}{n}}\cdot(-p)e^{-\gamma p}\prod_{n=1}^{\infty}\left(1-\frac{p}{n}\right)e^{\frac{p}{n}} = -p^2\prod_{n=1}^{\infty}\left(1-\frac{p^2}{n^2}\right)$$

한편 감마함수의 성질에 의해 $\Gamma(1-p) = -p\Gamma(-p)$이므로

$$\frac{1}{\Gamma(1-p)\Gamma(p)} = p\prod_{n=1}^{\infty}\left(1-\frac{p^2}{n^2}\right)$$

이고 바이어슈트라스 분해 정리에 의해

$$\sin z = z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right)$$

임을 이용하면

$$\frac{1}{\Gamma(1-p)\Gamma(p)} = p\prod_{n=1}^{\infty}\left(1-\frac{p^2}{n^2}\right) = \frac{\sin(\pi p)}{\pi}$$

얻고 증명이 완료되었다. ■

$$\therefore \int_0^{\frac{\pi}{2}}\sqrt[n]{\tan x}\,dx = \int_0^{\infty}\frac{t^{\frac{1}{n}}}{1+t^2}dt = \frac{\pi}{2}\sec\left(\frac{\pi}{2n}\right) \;\blacksquare\; (2\le n\in\mathbb{N})$$